Homework Set 3, Math 502 Spring 2013

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These exercises are due Wednesday, March 6.

1. Let μ be Lebesgue measure on $\mathbb{R}^n$. Let $f \in L^p(\mu)$, $1 \le p < \infty$. Let $\epsilon > 0$.

(a) Show that there exists a compact set $K \subset \mathbb{R}^n$ and a continuous function g supported on K such that $\|f - g\|_p < \epsilon/2$.

(b) Using the Stone-Wierstrass Theorem, show that there is a polynomial h in $x_1, \dots, x_n$ with rational coefficients such that

$$\left(\int_K |h(x) - g(x)|^p \mathrm{d}\mu \right)^{1/p} < \epsilon/2 \ .$$

(c) Show that $L^p(\mu)$, $1 \le p < \infty$ is *separable*; i.e., that there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ that is dense in $L^p(\mu)$.

(d) $L^\infty(\mu)$ to be the set of (equivalence classes of) measurable functions f on $\mathbb{R}^n$ such that for some $a < \infty$, $\mu(\{x \ : \ |f(x)| > a\}) = 0$. Define $\|f\|_\infty$ to be the infimum of all such a. Show that $\|\cdot\|_\infty$ is a norm, and that equipped with this norm, $L^\infty(\mu)$ is a complete metric space, but that it is not separable.

For the next problem, recall the reverse Hölder inequality:

0.1 LEMMA. *Let $0 < r < 1$ and let $s = r/(r-1) < 0$. Then for all n and all $a_j \ge 0$, $b_j > 0$, $i = j, \dots, n$,*

$$\sum_{j=1}^n a_j b_j \ge \left(\sum_{j=1}^n a_j^r \right)^{1/r} \left(\sum_{j=1}^n b_j^s \right)^{1/s} \tag{0.1}$$

2. Let $(X, \mathcal{F}, \mu)$ be a measure space, and for $1 < p \le 2$.

(a) Let $f, g \in L^p(\mu)$ with $f \ne g$. Suppose first that f and g are simple functions of the form

$$f(x) = \sum_{j=1}^N w_j 1_{A_j}(x) \qquad \text{and} \qquad g(x) = \sum_{j=1}^N z_j 1_{A_j}(x)$$

where each $w_j, z_j \in \mathbb{C}$, each $A - j$ is measurable, and $w_j z_j^*$ is not real for any j. (Here 1_A is the indicator function of A.) Show, using the lemma, that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \|f + tg\|_p^2 \ge 2(p-1)\|g\|_p^2 \ .$$

(b) Let $\psi(t)$ be a real valued function on $\mathbb{R}$ such that $\psi''(t) \geq 2c$, where the primes denote derivatives, and $c \in \mathbb{R}$. Define φ by

$$\varphi(t) = \psi(t) + ct(1-t) \ .$$

Show that φ is convex, and that $\varphi(0) = \psi(0)$ and $\varphi(1) = \psi(1)$. Show also that

$$\psi(1/2) + c/4 \leq \frac{\psi(0) + \psi(1)}{2} \ .$$

(c) Combine parts **(a)** and **(b)** to show that

$$\|f + g/2\|_p^2 + \frac{(p-1)}{4}\|g\|_p^2 \leq \frac{\|f\|_p^2 + \|f+g\|_p^2}{2} \ .$$

(d) Remove the simple-function approximation to show that for all unit vectors $u, v\ inL^(\mu)$,

$$\left\|\frac{u+v}{2}\right\|_p^2 + (p-1)\left\|\frac{u-v}{2}\right\|_p^2 \leq 1 \ .$$