Homework Set 2, Math 502 Spring 2013

Eric A. Carlen[1]
Rutgers University

February 9, 2013

These exercises are due Monday, Feb 18.

1. Let μ denote Lebesgue measure on $\mathbb{R}^n$. For $f\ inL^p(\mu)$, and $y \in \mathbb{R}^n$, define $\tau_y f$ by

$$\tau_y f(x) = f(x-y)\ .$$

Prove that

$$\lim_{y\to 0} \|\tau_y f - f\|_p = 0$$

for all $f \in L^p(\mu)$.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space, and for $1 \le p < \infty$ define $C \subset L^p(\mu)$ by

$$C := \{\ f \in L^p(\mu)\ :\ 0 \le f(x) \le 1 \qquad a.e.\ \}\ .$$

For which values of p, if any, is C closed in the L^p norm topology? Justify your answer.

3. Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $1 \le p < q < r \le \infty$. Suppose that $f \in L^p$ and $f \in L^r$. Then, as show in Propositon 6.10 in Folland, $f \in L^q$, and $\|f\|_q$ is bounded above by a certain geometric mean of $\|f\|_p$ and $\|f\|_r$. What about a lower bound? Show that for all $\epsilon > 0$ and all $q \in (p, r)$, there exists an f with $\|f\|_p = \|f\|_r = 1$ and $\|f\|_q < \epsilon$.

4. Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $1 \le p < q < r \le \infty$. Suppose that $f \in L^p$ and $f \in L^r$. Then, as show in Propositon 6.10 in Folland, $f \in L^q$. As the exercise above shows, there is in general now lower bound on $\|f\|_q$ given $\|f\|_p$ and $\|f\|_r$. However, if we have such a lower bound on $\|f\|_q$, then this implies a *point wise* lower bound on $|f|$ on a sizable set. More precisely, fix $\epsilon > 0$ and define

$$A_\epsilon = \{\ x\ :\ |f(x)| > \epsilon\ \}\ .$$

Show that

$$\|f\|_q^q \le \epsilon^{q-p}\|f\|_p^p + (\mu(A_\epsilon))^{(r-q)/r}\|f\|_r^q\ ,$$

and hence,

$$\epsilon^{q-p} < \frac{1}{2}\frac{\|f\|_q^q}{\|f\|_p^p} \quad \Rightarrow \quad \mu(A_\epsilon) \ge \left(\frac{\|f\|_q}{\|f\|_r}\right)^{qr/(r-q)}\ .$$

Finally, show that for each $\delta > 0$, there exists an $f \in L^p \cap L^r$ with $\|f\|_p = \|f\|_r = 1$ and

$$\mu(A_\epsilon) < \delta\ ,$$

so that the lower bound on $\|f\|_q$ with "q in the middle" was essential for having a lower bound on $\mu(A_\epsilon)$.

5. For $1 < p < \infty$, let $q = p/(p-1)$. Suppose $\{f_n\}$ is a sequence of unit vectors in L^p on some measure space $(X, \mathcal{F}, \mu)$. Suppose that g is a unit vector in L^q for the same measure space. Finally, suppose that both f and g are non-negative.

Show that if

$$\lim_{n\to\infty} \int_X f_n g \mathrm{d}\mu = 1 \ ,$$

then

$$\lim_{n\to\infty} \|f_n - g^{p-1}\|_p = 0 \ .$$

Is this true for $p = 1$ or $p = \infty$? What can you say if we drop the requirement that f and g be non-negative?