Take Home Final Exam for Math 502

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1. Recall the Projection Lemma, which says that in a uniformly convex Banach space, every closed convex set contains a unique element of minimal norm. The first two parts of this problem concerns what can go wrong if uniform convexity is not present.

(a) Let $\mathcal{C}$ be the space of continuous functions on $[0,1]$ equipped with the uniform norm. Let

$$K = \left\{ f \in \mathcal{C} \ : \ \int_0^{1/2} f(t)\mathrm{d}t - \int_{1/2}^1 f(t)\mathrm{d}t = 1 \right\} \ .$$

Show that K is closed and convex, but contains no element of minimal norm.

(b) Let μ be Lebesgue measure on $[0,1]$, and let

$$K = \left\{ f \in L^1(\mu) \ : \ \int_{[0,1]} f\mathrm{d}\mu = 1 \right\} \ .$$

Show that K is closed and convex, but contains infinitely many elements of minimal norm.

(c) Let μ be Lebesgue measure on $\mathbb{R}$. Let $1 < p < q < \infty$. As in Problem 4 on page 196 of Folland, define, the norm $\|\cdot\|$ on $L^p(\mu) + L^q(\mu)$ by

$$\|f\| = \inf \left\{ \|g\|_p + \|h\|_q \ : \ f = g + h, \ g \in L^p(\mu) \ , h \in L^q(\mu) \ \right\} \ .$$

Is $L^p(\mu) + L^q(\mu)$ uniformly convex in this norm?

2. **(a)** Let L be a bounded linear functional on a subspace $\mathcal{V}$ of some Hilbert space $\mathcal{H}$. Prove that L has a *unique* norm preserving extension from $\mathcal{V}$ to all of $\mathcal{H}$, and that this extension is zero on $\mathcal{V}^\perp$, the subspace of $\mathcal{H}$ that is orthogonal to $\mathcal{V}$.

(b) Construct a linear functional L on some subspace of $L^1(\mu)$ that has infinitely many norm preserving extensions to $L^1(\mu)$.

3. Let $\mathcal{C}$ be the space of continuous real valued functions on $[0,1]$ with the uniform norm. For $n \in \mathbb{N}$, define X_n to be the subset of $\mathcal{C}$ consisting of functions f such that for *some* $t \in [0,1]$,

$$|f(t) - f(s)| \le n|t-s|$$

for all $s \in [0,1]$.

(a) Show that for each fixed n, each open set in $\mathcal{C}$ contains an open set that does not intersect X_n.

(b) Show that this implies the existence of a dense G_δ (countable intersection of open sets) consisting entirely of nowhere differentiable functions.

4. Let μ and ν be finite Radon measures on $\mathbb{R}^n$. For $f \in \mathcal{C}_c(\mathbb{R}^n)$, define

$$L(f) = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x+y)\mathrm{d}\mu(x)\mathrm{d}\nu(y) \ .$$

(a) Show that there is a finite Radon measure λ on $\mathbb{R}^n$ so that

$$L(f) \ \int_{\mathbb{R}^n} f(x)\mathrm{d}\lambda(x)$$

for all $f \in \mathcal{C}_c(\mathbb{R}^n)$.

(b) Show that if μ and ν are absolutely continuous with respect to Lebesgue measure, so is λ, and give a formula for the Radon-Nikodym derivative of λ with respect to Lebesgue measure in terms of those for μ and ν.