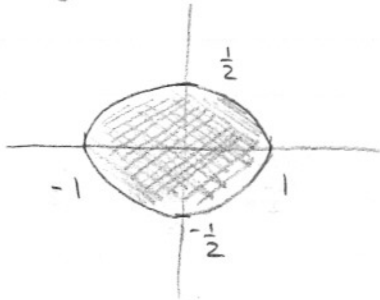
14.8 #19

Find extreme values of $f(x,y)=e^{-xy}$ on region $x^2+4y^2 \le 1$

Let $g(x,y)=x^2+4y^2$. The region $x^2+4y^2 \le 1$ is an elliptical disk:



To find relative maxima, we use methods from 14.7:

$$\left\{ \begin{array}{l} f_x = -ye^{-xy} = 0 \\ f_y = -xe^{-xy} = 0 \end{array} \right\} \Longrightarrow x=y=0$$

so $(0,0)$ is the only point with a level tangent plane, i.e. the only critical point on the interior.

We need to search the edge of the region, $x^2+4y^2=1$. Here we use 14.8, Lagrange Multipliers.

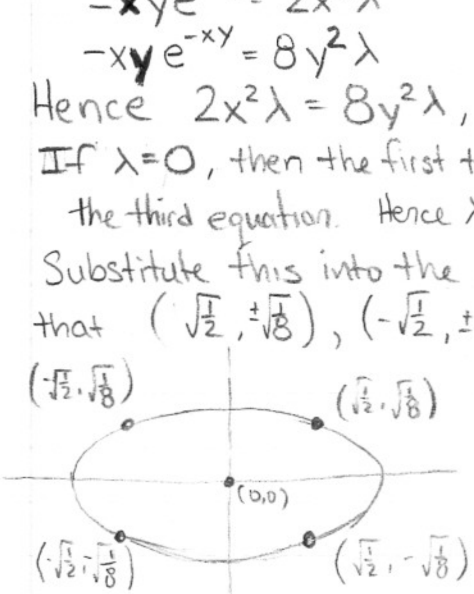
We solve
$$\left\{ \begin{array}{l} f_x = -ye^{-xy} = \lambda g_x = \lambda \cdot 2x \\ f_y = -xe^{-xy} = \lambda g_y = \lambda \cdot 8y \\ x^2+4y^2=1 \end{array} \right.$$

From the first two equations, we can deduce

$$-\mathbf{x}ye^{-xy} = 2x^{\mathbf{2}}\lambda$$
$$-x\mathbf{y}e^{-xy} = 8y^2\lambda$$

Hence $2x^2\lambda = 8y^2\lambda$, so $x^2\lambda = 4y^2\lambda$

If $\lambda=0$, then the first two equations tell us that $x=y=0$, contradicting the third equation. Hence $\lambda \ne 0$, so we can divide it out to get $x^2=4y^2$. Substitute this into the third equation to get $2x^2=1$, implying that $\left(\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{8}}\right)$, $\left(-\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{8}}\right)$ are all critical points.



Compare: $f(0,0)=1$

$f\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{8}}\right) = f\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{8}}\right) = e^{-\frac{1}{4}} \approx 0.779$ MIN

$f\left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{8}}\right) = f\left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{8}}\right) = e^{\frac{1}{4}} \approx 1.284$ MAX

Hence, the maximum value for f on the region is $e^{\frac{1}{4}}$ and the minimum value for f on the region is $e^{-\frac{1}{4}}$.

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