Math 103 – Practice Exam I
SOLUTIONS

Complete all questions. You may use scrap paper and a calculator, but no notes or textbook. You must show all work to qualify for full credit. **No work = no credit**. Please write neatly. If I can't read it, I can't grade it, and you can't get points.

1. (25 Pts) An election takes place amongst four candidates. After the ballots are collected, we get the following preference schedule:

	13	5	8	9
1st	A	B	C	C
2nd	B	A	B	A
3rd	D	D	A	B
4th	C	C	D	D

(a) Find the **winner** using the **method of pairwise comparisons**.
Winner in bold: **A** vs B, **A** vs C, **A** vs D, **B** vs C, **B** vs D, C vs **D**
Score: A: 3, B: 2, C:0, D:1.
A wins by pairwise comparisons.

(b) Find the **winner** using the **plurality with elimination**.
Since there are 35 ballots, you'll need 18 votes for a majority

Round 1	Round 2	Round 3
A: 13	A: 13	A: 18
B: 5	B:5	(B out)
C: 17	C: 17	C: 17
D: 0	(D out)	(D out)
Elim D	Elim B	A has majority

A wins by plurality with elimination.

(c) Is there a **majority candidate**? If so, who is it? If not, why not?
There is no majority candidate. A candidate would need at least 18 first-place votes to be a majority candidate, and the most anyone has is 17.

(d) Is there a **Condorcet candidate**? If so, who is it? If not, why not?
Yes, A is a Condorcet candidate. It beats each other candidate in a head-to-head comparison.

(e) Find the **ranking** of the candidates using the **extended Borda count**.
Scores: A: 110, B: 101, C: 86, D:53
Ranking: A, B, C, D

2. (20 Pts)
(a) Find the **Banzhaf power distribution** for the weighted voting system [12: 9, 5, 5, 1] (Hint: you should get six winning coalitions)

Winning Coalition	Critical Players
$\{P_1, P_2, P_3, P_4\}$	P_1
$\{P_1, P_3, P_4\}$	P_1, P_3
$\{P_1, P_2, P_4\}$	P_1, P_2
$\{P_1, P_2, P_3\}$	P_1
$\{P_1, P_2\}$	P_1, P_2
$\{P_1, P_3\}$	P_1, P_3

Then we get B_1: 6, B_2: 2, B_3: 2, B_4: 0, so then T=10. Therefore the Banzhaf Power Distribution is: β_1=6/10=60%, β_2= 2/10=20%, β_3= 2/10=20%, β_4=0/10=0%

(b) Is any player a **dictator**? If so, who? Justify your answer.
No player is a dictator because no player has more than the quota in votes.

(c) Is any player a **dummy**? If so, who? Justify your answer.
P_4 is a dummy. They have 0% power.

(d) Does any player have **veto power**? If so, who? Justify your answer.
P_1 has veto power. They are critical in every winning coalition. Alternately, you could notice that the other three players do not enough votes (viz. the quota) to pass any motion by themselves.

For (e) and (f), use the following scenario: The city government of Coolsville is trying to implement the strong-mayor system. They set up a city council of three members and a mayor. If two of the three council members approve a motion, the mayor has the power to veto the motion. However, if all three of the council members vote to pass a motion then the mayor cannot veto it.

(e) List all sequential coalitions in the table below. Circle the pivotal player in each sequential coalition. (All boxes should be filled. If you're careful when filling in the table, part (f) will be easier)
Let's let the council members be A, B, and C, and let the mayor be M. Pivotal player in boldface

<M,A,**B**,C>	<A,M,**B**,C>	<A,B,**M**,C>	<A,B,**C**,M>
<M,A,**C**,B>	<A,M,**C**,B>	<A,C,**M**,B>	<A,C,**B**,M>
<M,B,**A**,C>	<B,M,**A**,C>	<B,A,**M**,C>	<B,A,**C**,M>
<M,B,**C**,A>	<B,M,**C**,A>	<B,C,**M**,A>	<B,C,**A**,M>
<M,C,**A**,B>	<C,M,**A**,B>	<C,A,**M**,B>	<C,A,**B**,M>
<M,C,**B**,A>	<C,M,**B**,A>	<C,B,**M**,A>	<C,B,**A**,M>

You need either two council members and the mayor, or three council members to pass a motion. Thus it's always the third player in a sequential coalition that is pivotal.

f) Compute the **Shapley-Shubik power distribution** for the system. Does the mayor really have any more power than the council members? Each player is pivotal 6 times out of the 24 possible

times. Thus each player has Shapley-Shubik power 25%, and the mayor isn't any more powerful than the council members.

3. (12 pts) Tony, Carmela, Janice, and Bobby are playing Monopoly when Bobby quits over the introduction of the "Free Parking rule." The other three have already divided his money evenly amongst themselves, but now they must divide Bobby's three acquired properties (Reading Railroad, Virginia Avenue, Marven Gardens, and Park Place) fairly. They decide to use the **method of sealed bids**. Below is a table listing each player's bid.

	Tony	Carmela	Janice
Reading Railroad	$250	$300	$160
Virginia Avenue	$260	$130	$190
Marven Gardens	$280	$330	$300
Park Place	$500	$470	$460
Total	1290	1230	1110
Fair share	430	410	370

(a) In the first allocation, who gets what properties (money is covered in (c))?
Tony: Virginia Ave, Park Place
Carmela: Reading Railroad, Marven Gardens
Janice: None

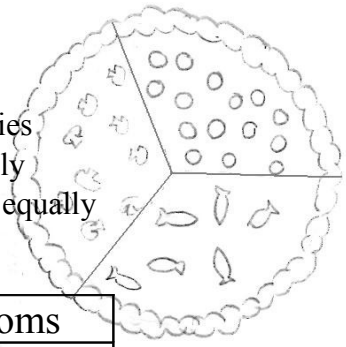
(b) Fill in the "total" and "fair share" rows of the table.
See table.
(c) In the first allocation, how much money does each player lose or receive?
Tony pays ($260 + $500) - $430 = $330
Tony pays ($300 + $330) - $410 = $220
Janice gets $370

(d) After the first allocation, how much of a surplus is there (i.e. how much is "in the pot")?
4330+$220-$370 =$ 180

(e) What is each player's share of the surplus?
$180 ÷ 3 = $60

(f) What is the final allocation of properties and money? (Be sure to mention whether a player is losing or gaining money).
Tony gets Virginia Ave and Park Place, and pays $330-$60=$270
Carmela gets Reading Railroad, Marven Gardens, and pays $220-$60=$160
Janice gains $370+$60=$430

4. (18 Pts) Leonardo, Donatello, Michelangelo, and Raphael are dividing a $12 pizza. It's up to you whether these are the Teenage Mutant Ninja Turtles or the Renaissance artists. The pizza is divided evenly into Pepperoni, Anchovies, and Mushrooms sections (See picture).



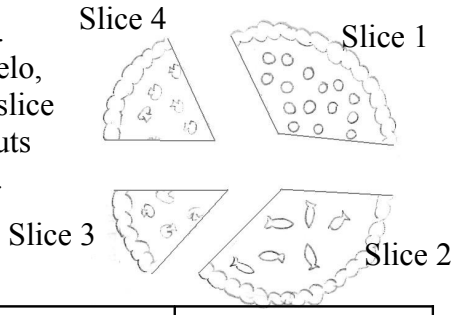
The hungry turtles/artists have the following preferences:

- Leonardo's values are as in the chart in part (a)
- Donatello is a vegetarian and will not eat pepperoni or anchovies
- Michelangelo is a party dude and likes all kinds of pizza equally
- Raphael hates anchovies, and likes pepperoni and mushrooms equally

(a) Complete the chart below, based on the preferences above.

	Pepperoni	Anchovies	Mushrooms
Leonardo	3	3	6
Donatello	0	0	12
Michelangelo	4	4	4
Raphael	6	0	6

They decide to cut up the pizza using the **Lone Divider Method**. Leonardo is chosen as the divider, leaving Donatello, Michelangelo, and Raphael to be the choosers. Leonardo cuts the piece so that slice 1 is the pepperoni portion, slice 2 is the anchovies portion, and cuts the mushroom portion in half to form slices 3 and 4 (see picture).



(b) What is each slice worth (in $$) to each player?

	Slice 1	Slice 2	Slice 3	Slice 4
Leonardo	3	3	3	3
Donatello	0	0	6	6
Michelangelo	4	4	2	2
Raphael	6	0	3	3

(e) Did Leonardo **cut consistently with his value system**? Justify your answer.
Yes. Each piece is worth a fair share ($3).

(d) What is the **bid list** for each chooser?
D: {Slice 3, Slice 4}
M: {Slice 1, Slice 2}
R: {Slice 1, Slice 3, Slice 4}
(Leonardo doesn't have a bid list, as he is not a chooser)

(e) What would be a **fair distribution** for the four slices of pizza?
There are many answers. One of them is:

Leo: 4
Don: 3
Mike:2
Raph: 1

5. (25 pts) The Galactic Parliament has 11 seats, to be distributed among the three states (Andros, Baxteros, and Kristenos) in proportion to their population. Their populations (in billions) are summarized below

State	Andros	Baxteros	Kristenos
Population	54	243	703

(a) What is the **standard divisor** (round to two decimal places)? How can it be interpreted?
Total population is 1000.
SD=1000 ÷ 11 = 90.91
For every 90.91 (billion) people in your state, you should get 1 seat in Parliament.

(b) Apportion the seats using **Hamilton's method**.

State	Population	Quotas	Lower Quota	Adustment	Hamilton's Apportionment
A	54	0.59	0		0
B	243	2.67	2	+1	3
K	703	7.73	7	+1	8
Total	1000		9		11

(There is nothing in the normal rules that says that every state must get at least 1 seat)

(c) A new state, Wicketeria, joins the Galactic Alliance. Its population is 580, so 7 seats are added to Parliament. What is the new apportionment using **Hamilton's Method**? (Don't forget to recalculate the standard divisor)
New SD=1580 ÷ 18=87.78

State	Population	Quotas	Lower Quota	Adjustment	Hamilton's Apportionment
A	54	0.62	0	+1	1
B	243	2.77	2	+1	3
K	703	8.01	8		8
W	580	6.61	6		6
Total	1580		16		18

(d) Did any **paradoxes** occur? If so, state which paradox and justify your answer.
The new-states paradox occurred. A new state was added, and even though new seats were added for it, its addition affected the apportionment of the other states.

(e) Citing the above paradox, Senator Xaphod Beeblebrox (from Kristenos) calls for a new apportionment scheme, which he calls the Beeblebrox Method. In a historical coincidence, it's identical to Jefferson's Method. Use **Jefferson's Method** to apportion the 18 seats to the four states. (Timesaver: either the number 80 or 90 will come in handy)

State	Population	Quota (D=80)	Lower Quota	Jefferson's Apportionment
A	54	0.68	0	0
B	243	3.04	3	3
K	703	8.79	8	8
W	580	7.25	7	7
Total	1580		18	18