to Sush...

HW #6 KEY for Math 356

4pts

1 (a) Prove that the G.F. for ptns with exactly j parts is $\dfrac{q^j}{(1-q)\cdots(1-q^j)}$.

$$q^j \times \frac{1}{(1-q)\cdots(1-q^j)} = q^j \times \text{G.F. for partitions with largest part} \leq j$$

$= q^j \times$ G.F. for partitions with at most j parts (by conjugation)

$= q^j \times$ (G.F. for sums of the form $a_1 + a_2 + \cdots + a_j$ where $0 \leq a_1 \leq a_2 \leq \cdots \leq a_j$ and each $a_i \in \mathbb{Z}$)

$$= q^{\overbrace{1+1+\cdots+1}^{j \text{ terms}}} \times (\quad " \quad)$$

= G.F. for sums of the form

$(1+a_1) + (1+a_2) + \cdots + (1+a_j)$ with a_i's as above

= G.F. for sums of the form

$b_1 + b_2 + \cdots + b_j$ where $b_i = 1 + a_i$ and a_i as above

so $1 \leq b_1 \leq b_2 \leq \cdots \leq b_j$

= G.F. for partitions with exactly j parts.

2pts

(b) $\displaystyle\sum_{j=0}^{\infty} \frac{q^j}{(1-q)\cdots(1-q^j)}$ is the G.F. for all partitions (by summing the expression from 1(a) over all j).

$\displaystyle\prod_{k=1}^{\infty} \frac{1}{1-q^k}$ is also the G.F. for all partitions (by Euler).

$\therefore$ result follows.

2 Let $R(n) = \#$ of ptns of n into parts which are distinct, nonconsecutive integers and

$S(n) = \#$ of ptns of n into parts congruent to 1 or 4 (mod 5)

$\boxed{\text{2pts}}$

(a) $$\sum_{n=0}^{\infty} S(n) q^n = \prod_{\substack{m \geq 1 \\ m \equiv 1,4 \pmod 5}} \left(1+q^m+q^{2m}+q^{3m}+\cdots\right)$$

$$= \prod_{\substack{m \geq 1 \\ m \equiv 1,4 \pmod 5}} \frac{1}{1-q^m} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}.$$

$\boxed{\text{4pts}}$

(b) $$\frac{q^{j^2}}{(1-q)\cdots(1-q^j)} = q^{j^2} \times \text{G.F. for ptns with largest part} \leq j$$

$= q^{j^2} \times$ G.F. for ptns with at most j parts

$= q^{j^2} \times$ (G.F. for sums of the form $a_1 + a_2 + \cdots + a_j$ with a_i's as in problem 1(a))

$= q^{1+3+5+\cdots+(2j-1)} \times (\quad " \quad)$

$=$ G.F. for sums of the form $(1+a_1) + (3+a_2) + \cdots + ((2j-1)+a_j)$

$=$ G.F. for sums of the form $b_1 + b_2 + \cdots + b_j$ where $b_i = (2i-1)+a_i$

$=$ G.F. for ptns $b_1 + b_2 + \cdots + b_j$ into exactly j parts where $b_i - b_{i-1} \geq 2$ for $i = 2, 3, \ldots, j$.

(c) $$\sum_{j=0}^{\infty} R(n) q^n = \sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)\cdots(1-q^j)}$$ by summing the result of 2(b) over all j

(d) Assuming the first RR identity

$$\sum_{n=0}^{\infty} R(n) q^n \overset{\text{by (c)}}{=} \sum_{j=0} \frac{q^{j^2}}{(1-q)\cdots(1-q^j)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})} \overset{\text{by (a)}}{=} \sum_{n=0}^{\infty} S(n) q^n.$$

Comparing coeffs of q^n in the extremes, we find $R(n) = S(n)$ for $n \in \mathbb{Z}_+$