EXPOSITORY NOTE: An Arithmetic Surface[1]

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In this note we work out a brutally explicit example of a compact (no cusps) arithmetic surface, by constructing a uniform (no unipotents) arithmetic ($\mathbb{Z}$-points) lattice, that is, a discrete $\mathbb{Q}$-subgroup $\Gamma < \mathrm{SL}(2,\mathbb{R})$ with finite co-volume.

For $\mathbf{x} = (x, y, z)$, let $Q(\mathbf{x})$ be the ternary quadratic form

$$Q(\mathbf{x}) = x^2 + y^2 - 3z^2 = \mathbf{x} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -3 \end{pmatrix} \mathbf{x}^t.$$

It is clearly indefinite (takes positive and negative values), and anisotropic over $\mathbb{Q}$. The latter means there are no $\mathbb{Q}$ points on the cone $Q = 0$ (it is enough to consider $\mathbb{Z}$ points (why?), and 3 is not the sum of two squares). The special orthogonal group $G = \mathrm{SO}_Q \cong \mathrm{SO}(2,1)$ preserving Q is the set

$$\begin{aligned} G &:= \left\{ g \in \mathrm{SL}(3,\mathbb{R}) : Q(\mathbf{x}g) = Q(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}^3 \right\} \\ &= \left\{ g \in \mathrm{SL}(3,\mathbb{R}) : g \begin{pmatrix} 1 & & \\ & 1 & \\ & & -3 \end{pmatrix} g^t = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -3 \end{pmatrix} \right\}. \end{aligned}$$

In what follows, we construct the spin representation of SO_Q, which is double-covered by $\mathrm{SL}(2,\mathbb{R})$. Consider symmetric matrices of the form

$$m_{\mathbf{x}} := \begin{pmatrix} z\sqrt{3} - y & x \\ x & z\sqrt{3} + y \end{pmatrix}.$$

These are cooked up to have the property that $\det(m_{\mathbf{x}}) = -Q(\mathbf{x})$. Clearly given $m_{\mathbf{x}}$, we can read off $\mathbf{x}$, e.g.:

$$\frac{1}{2\sqrt{3}} \operatorname{tr}(m_{\mathbf{x}}) = z. \tag{0.1}$$

Now, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, we have the action on $m_{\mathbf{x}}$ given by:

$$g \circ m_{\mathbf{x}} := g \cdot m_{\mathbf{x}} \cdot g^t$$

which is clearly also symmetric, and satisfies

$$\det(g \circ m_{\mathbf{x}}) = (ad - bc)^2 \det(m_{\mathbf{x}}),$$

meaning we can write $g \circ m_{\mathbf{x}}$ as $m_{\mathbf{x}'}$ for some $\mathbf{x}' = (x', y', z')$. It is straightforward to compute $\mathbf{x}'$:

$$\begin{aligned} m_{\mathbf{x}'} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} z\sqrt{3} - y & x \\ x & z\sqrt{3} + y \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} \\ &= \begin{pmatrix} -ya^2 + \sqrt{3}za^2 + 2bxa + b^2y + \sqrt{3}b^2z & bcx + adx - acy + bdy + \sqrt{3}acz + \sqrt{3}bdz \\ bcx + adx - acy + bdy + \sqrt{3}acz + \sqrt{3}bdz & -yc^2 + \sqrt{3}zc^2 + 2dxc + d^2y + \sqrt{3}d^2z \end{pmatrix}, \end{aligned}$$

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so we can read off x', y', z' as in (0.1), and we find that

$$\mathbf{x}' = (x,y,z)\cdot \begin{pmatrix} bc+ad & cd-ab & \frac{ab+cd}{\sqrt{3}} \\ bd-ac & \frac{1}{2}\left(a^2-b^2-c^2+d^2\right) & \frac{-a^2+b^2-c^2+d^2}{2\sqrt{3}} \\ \sqrt{3}(ac+bd) & \frac{1}{2}\sqrt{3}\left(-a^2-b^2+c^2+d^2\right) & \frac{1}{2}\left(a^2+b^2+c^2+d^2\right) \end{pmatrix}.$$

This of course means that the matrix above is an element of G, and hence we have cooked up a map $\iota : \mathrm{SL}(2,\mathbb{R}) \to G = \mathrm{SO}_Q$, sending

$$\iota : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{ad-bc} \begin{pmatrix} bc+ad & cd-ab & \frac{ab+cd}{\sqrt{3}} \\ bd-ac & \frac{1}{2}\left(a^2-b^2-c^2+d^2\right) & \frac{-a^2+b^2-c^2+d^2}{2\sqrt{3}} \\ \sqrt{3}(ac+bd) & \frac{1}{2}\sqrt{3}\left(-a^2-b^2+c^2+d^2\right) & \frac{1}{2}\left(a^2+b^2+c^2+d^2\right) \end{pmatrix}.$$

It's a double-cover because $-\mathbf{1}$ gets mapped to the same thing as $\mathbf{1}$ (so we could have used $\mathrm{PSL}(2,\mathbb{R})$).

This was all over $\mathbb{R}$. But in fact there are two $\mathbb{Q}$ structures of SL_2, one of which is the obvious $\mathrm{SL}_2(\mathbb{Q})$, and the other consists of norm 1 elements in a quaternion division algebra. It is the latter which leads to compact arithmetic surfaces, as follows.

Let I, J, K formally satisfy $I^2 = 3$, $J^2 = 3$, and $K = \frac{1}{3}IJ$, so $K^2 = -1$. Then form the quaternion

$$\mathbf{u} = a + bI + cJ + dK.$$

The norm is

$$N(\mathbf{u}) = \mathbf{u}\bar{\mathbf{u}} = a^2 - 3b^2 - 3c^2 + d^2.$$

Let D_Q^1 be the elements $\mathbf{u} \in D_Q$ with $N(\mathbf{u}) = 1$. A morphism $\rho : D_Q^1 \to G$ maps

$$\rho : \mathbf{u} \mapsto \begin{pmatrix} a^2-3b^2+3c^2-d^2 & 2ad+6bc & -2(ac+bd) \\ 6bc-2ad & a^2+3b^2-3c^2-d^2 & 2cd-2ab \\ 6bd-6ac & -6(ab+cd) & a^2+3b^2+3c^2+d^2 \end{pmatrix}.$$

One can check directly that

$$\rho(\mathbf{u}) \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & -3 \end{pmatrix} \cdot \rho(\mathbf{u})^t = N(\mathbf{u})^2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & -3 \end{pmatrix},$$

so $\rho(\mathbf{u}) \in G$ if $N(\mathbf{u}) = 1$. Then our co-compact lattice will come from the $\mathbb{Z}$-elements of D_Q^1.

What's the connection between the two morphisms ι and ρ? Quaternion division algebras can be realized as 2×2 matrices. Write

$$\begin{aligned} \mathbf{1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ I &= \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix} \\ J &= \begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix} \\ K &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

and check that $\mathbf{1}, I, J, K$ satisfy the above formal conditions. Then we can write $\mathbf{u}$ as

$$\mathbf{u} = a\mathbf{1} + bI + cJ + dK = \begin{pmatrix} a + \sqrt{3}b & -d - \sqrt{3}c \\ d - \sqrt{3}c & a - \sqrt{3}b \end{pmatrix}.$$

Obviously with the above representation we have $\det(\mathbf{u}) = a^2 - 3b^2 - 3c^2 + d = N(\mathbf{u})$, so if $\mathbf{u} \in D_Q^1$ has norm one, then it also lives in $\mathrm{SL}_2(\mathbb{R})$. That means we can apply ι, and in fact we have

$$\iota : \begin{pmatrix} a + \sqrt{3}b & -d - \sqrt{3}c \\ d - \sqrt{3}c & a - \sqrt{3}b \end{pmatrix} \mapsto \begin{pmatrix} a^2 - 3b^2 + 3c^2 - d^2 & 2ad + 6bc & -2(ac + bd) \\ 6bc - 2ad & a^2 + 3b^2 - 3c^2 - d^2 & 2cd - 2ab \\ 6bd - 6ac & -6(ab + cd) & a^2 + 3b^2 + 3c^2 + d^2 \end{pmatrix},$$

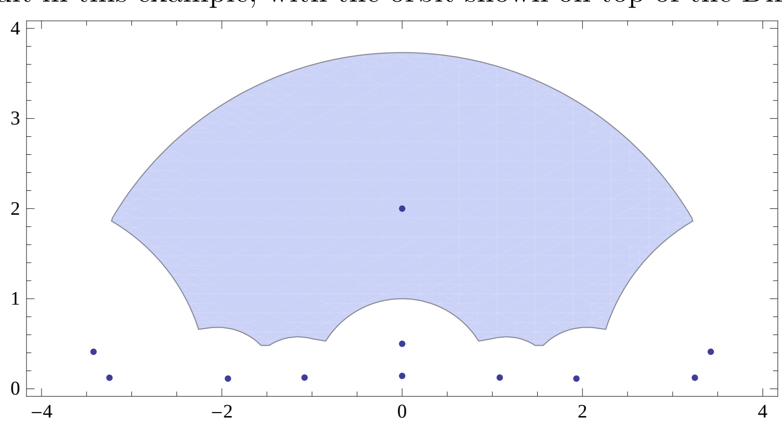
which is our old friend $\rho(\mathbf{u})$. (When we first introduced it, we pulled it out of thin air, but now its role is clear.) To get our discrete group Γ, we simply insist that $a, b, c, d \in \mathbb{Z}$.

Summarizing, we see that what we really want is elements $\alpha = a + \sqrt{3}b$, and $\beta = d + \sqrt{3}c$ in the ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{3}]$ of the number field $K = \mathbb{Q}[\sqrt{3}]$. We put these in a matrix of the form:

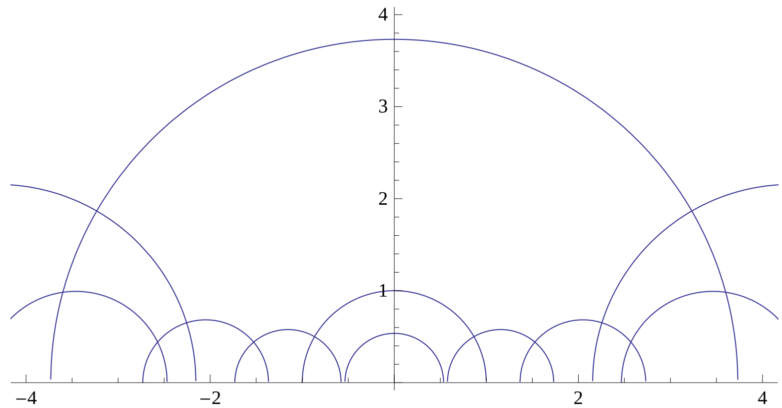
$$M_{\alpha,\beta} := \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix},$$

and ask that $\det M_{\alpha,\beta} = N(\alpha) + N(\beta) = a^2 - 3b^2 + d^2 - 3c^2 = 1$.

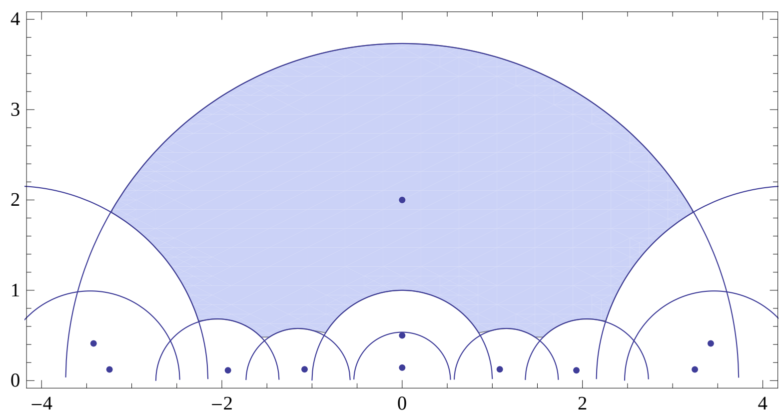
It is easy enough to do a brute search for small elements $\gamma \in \Gamma$, take the orbit under these elements of some fixed base point, say $z_0 = 2i$, and construct the corresponding Dirichlet domain. The result in this example, with the orbit shown on top of the Dirichlet domain, is:



Note that for any point w in the orbit of $z_0 = 2i$ under Γ, one can draw the set of all points equidistant to w and z_0. This will be a geodesic, and those corresponding to the closest points will determine the bounding geodesics for the Dirichlet domain. Here they are in this case:



And the two pictures overlapped:



The group elements used in the calculation above were:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 2\left(1+\sqrt{3}\right) \\ 2\left(-1+\sqrt{3}\right) & -3 \end{pmatrix}, \begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & -2 \end{pmatrix}, \begin{pmatrix} -2 & 3+2\sqrt{3} \\ -3+2\sqrt{3} & -2 \end{pmatrix},$$

$$\begin{pmatrix} -3 & 2\left(1+\sqrt{3}\right) \\ 2\left(-1+\sqrt{3}\right) & -3 \end{pmatrix}, \begin{pmatrix} -3-2\sqrt{3} & -2 \\ 2 & -3+2\sqrt{3} \end{pmatrix}, \begin{pmatrix} 2-\sqrt{3} & 0 \\ 0 & 2+\sqrt{3} \end{pmatrix},$$

$$\begin{pmatrix} 0 & -2-\sqrt{3} \\ 2-\sqrt{3} & 0 \end{pmatrix}, \begin{pmatrix} -3-2\sqrt{3} & 2 \\ -2 & -3+2\sqrt{3} \end{pmatrix}, \begin{pmatrix} -3 & -2\left(1+\sqrt{3}\right) \\ 2-2\sqrt{3} & -3 \end{pmatrix},$$

$$\begin{pmatrix} -2 & -3-2\sqrt{3} \\ 3-2\sqrt{3} & -2 \end{pmatrix}, \begin{pmatrix} -2 & -\sqrt{3} \\ -\sqrt{3} & -2 \end{pmatrix}$$

Of course here we see the great advantage of listing these elements in their more natural structure, that of a quaternion division algebra: $u = a\mathbf{1} + bI + cJ + dK$, where

a	b	c	d
0	0	0	1
-3	0	-2	-2
-2	0	-1	0
-2	0	-2	-3
-3	0	-2	-2
-3	-2	0	2
2	-1	0	0
0	0	1	2
-3	-2	0	-2
-3	0	2	2
-2	0	2	3
-2	0	1	0