Alex V. Kontorovich

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1 Added 10/26/21:

These notes are from a research project supervised by John Nash, starting around 2001/2002. He had secured funding through the NSF, namely grant SES-0001711, and used it to hire research assistants. The notes below were written around 2003/2004 to document what had been done on the project. Nash's paper on these investigations eventually appeared in: Nash, John F., Jr., "The agencies method for modeling coalitions and cooperation in games." Int. Game Theory Rev. 10 (2008), no. 4, 539–564.

2 Introduction

There are two stages and three players in the game. In the first stage, two players join into a coalition, with one representing the other, while the third player is left by himself. In the second stage, all three players join into one coalition with either the third player heading the group, or the representative from the previous stage leading the entire pack.

2.1 Notation and Defined Relations

The players are denoted by $P1$, $P2$, and $P3$. When $P1$ accepts $P2$ to be his agent, we denote the coalition by $P21$, and from this point on, $P2$ will represent $P1$. Write Pi for a generic player or coalition, where i may be one or two numbers. To denote the payoff vector by $UBjRk$, where the coalition Pj represents Pk (Utility given By j Representing k). When we want individual components of the vector, we denote them by $u_i b_j r_k$ ($u$tility of i given by j $r$epresenting k), so that $UBjRk = (u_1 b_j r_k, u_2 b_j r_k, u_3 b_j r_k)$. Some examples of this:

1. The expected utility for $P3$ if he allows $P2$ to represent him is $u_3 b_2 r_3$.
2. When $P3$ accepts the coalition $P21$ to represent him, the payoff vector is $UB21R3$. This should be distinguished from the case
3. when the coalition $P13$ accepts $P2$ as their agent. In this case, the vector is $UB2R13$.

The probability that a player Pi accepts Pk is denoted by $a_i f_k$ (probability of $a$cceptance of i $f$or k). E.g. In the first round, the probability that $P1$ chooses $P2$ to represent him is $a_1 f_2$. We denote the probability with which coalition Pij accepts player Pk by $a_{ij} = a_{ij} f_k$, the default being the shorter form. Similarly, we write $af_{ij} = a_k f_{ij}$ for the reverse scenario.

For the first round, instead of $a_i f_i$, we write n_i (no acceptance) with the obvious rule that

$$a_i f_j + a_i f_k + n_i = 1, \tag{1}$$

where $j \neq k \neq i$.

Each player and coalition has a demand. It is a strategic variable, chosen to optimize profit based on equilibrium conditions which we will solve in Section 2. In the first stage, every player is on his own, so player Pi demands the same amount from both of the other players. We denote this demand by d_i. For the second stage, we write d_if_j for the $d$emand of player or coalition Pi $f$or player or coalition Pj, with the same shorthand as in the case a_if_j. So player $P3$ may have a different demand in the first stage, d_3, as he does for the coalition $P21$ (in the second stage) – $df_{21} = d_3f_{21}$. And his demand also differs depending on the coalition with which he is dealing, so df_{21} is not necessarily equal to df_{12}.

In the first stage, if no player chooses to have another represent him, the game is allowed to repeat with probability $(1 - \varepsilon_4)$. If the game is not allowed to repeat, it ends with a zero payoff for each player. If more than one player asks to be represented, a uniform random event decides which player is represented. So at the end of stage one, either the players all have zero payoff, or one player is alone and another is leading his two-man coalition.

A similar procedure occurs in the second stage. Should neither the coalition nor the lone player want to accept the other as the representative, the game ends with probability ε_5. If this occurs, the lone player, Pi, receives a zero payoff, and the other two share an amount $b_i \in [0,1]$. E.g. if coalition $P12$ does not accept $P3$ and vice versa, then with probability ε_5, the payoff vector is $\left(\frac{b_3}{2}, \frac{b_3}{2}, 0\right)$. For simplicity of notation, we will denote the last vector by $\frac{b_3}{2}\delta_3$, where δ_i has a 0 in the ith component of the vector, and 1 in the other two. E.g. $\delta_3 = (1,1,0)$.

With probability $(1 - \varepsilon_5)$, the second stage is allowed to repeat, giving the players another chance to join together.

If there is an acceptance in the second stage, and all three players join a coalition, and are awarded the full amount, 1. The player leading the coalition then divides the payoff among the members of his coalition. E.g. if $P2$ accepts $P1$ in the first round and $P12$ accepts $P3$ in the second, then player $P3$ determines the payoff vector, $UB3R12$. Since the total amount given to $P3$ is 1, it is clear that we must have $u_1b_3r_{12} + u_2b_3r_{12} + u_3b_3r_{12} = 1$. The same goes for any three-man coalition – the sum of the entries of the payoff vector is 1. Since the final payoff is determined by the lead player, it becomes a strategic variable, and has an equilibrium condition which we solve for in Section 2.

In the first stage, the probabilities are related to the demands and utilities in the following way. For players Pi and Pj, define

$$A_iF_j = \exp\left(\frac{u_ib_jr_i - d_i}{\varepsilon_3}\right), \tag{2}$$

and then let

$$a_if_j = \frac{A_iF_j}{1 + A_iF_j + A_iF_k}, \tag{3}$$

with $i \neq j \neq k$. The constant, ε_3, enters, determining the sensitivity of the demand. If ε_3 is close to zero, the slightest difference between the utility given and amount demanded drives the probability to zero or one, if the difference is negative or positive, respectively. The bigger ε_3, the less this difference is felt in the overall probability. It is easy to verify that $n_i = \frac{1}{1+A_iF_j+A_iF_k}$, and so $a_if_j + a_if_k + n_i = 1$. E.g. in stage one, the probability that player $P1$ chooses to accept player

$P2$ as his agent is:

$$\begin{aligned} a_1 f_2 &= \frac{\exp\left(\dfrac{u_1 b_2 r_1 - d_1}{\varepsilon_3}\right)}{1 + \exp\left(\dfrac{u_1 b_2 r_1 - d_1}{\varepsilon_3}\right) + \exp\left(\dfrac{u_1 b_3 r_1 - d_1}{\varepsilon_3}\right)} \\ &= \frac{\exp\left(\dfrac{u_1 b_2 r_1}{\varepsilon_3}\right)}{\exp\left(\dfrac{d_1}{\varepsilon_3}\right) + \exp\left(\dfrac{u_1 b_2 r_1}{\varepsilon_3}\right) + \exp\left(\dfrac{u_1 b_3 r_1}{\varepsilon_3}\right)}. \end{aligned}$$

In the second stage, again a similar situation occurs. For player Pi and coalition Pjk, define

$$AF_{jk} = A_i F_{jk} = \exp\left(\frac{u_i b_{jk} r_i - df_{jk}}{\varepsilon_3}\right), \tag{4}$$

and the formula for this stage is simply:

$$af_{jk} = \frac{AF_{jk}}{1 + AF_{jk}}. \tag{5}$$

For coalition Pij and player Pk, we have:

$$A_{ij} = A_{ij} F_k = \exp\left(\frac{u_i b_k r_{ij} - d_{ij}}{\varepsilon_3}\right), \tag{6}$$

and again:

$$a_{ij} = \frac{A_{ij}}{1 + A_{ij}}. \tag{7}$$

The variables of the form $a_i f_j$ denote a player's choice within the game. To denote an actual outcome, we need another notation. If a situation has payoff UBX, we write the probability of the event as $p\{X\}$. E.g. the utility vector associated with $P2$ accepting $P1$ is $UB1R2$, and so the probability of $P2$ succeeding in accepting $P1$ is $p\{1R2\}$. (In MATHEMATICA, this is simply $p12$.)

This is all of the notation we will be using. Now we take our assumptions and gather them together to derive formulaic relations of defined quantities.

2.2 Derived Relations

Say we have come to stage two, and in the first stage, player $P1$ accepted $P2$ to be his representative. This is associated with the payoff vector $UB2R1$, for which we will now try to find a formula. So we are looking at stage two, and we have the coalition $P21$ and the player $P3$. There are four possible combinations:

1. $P21$ chooses to accept $P3$, and $P3$ does not accept $P21$. This event has probability $(a_{21})(1 - af_{21})$, and the payoff vector is $UB3R21$.

2. $P21$ chooses not to accept $P3$, but $P3$ does accept $P21$. This event has probability $(1 - a_{21})(af_{21})$, and the payoff vector is $UB21R3$.

3. Both players accept the other as representatives. The event has probability $(a_{21}f_3)(a_3f_{21})$. From here, a fair coin is flipped to determine the outcome, so the expected payoff is simply $\frac{1}{2}(UB21R3 + UB3R21)$, where this is regular vector addition and scalar multiplication.

4. Neither player accepts the other. The event has probability $(1-a_{21})(1-af_{21})$. From here, as described above, we either repeat the round with probability $(1-\varepsilon_5)$ (this has expected utility $UB2R1$, exactly the value we're solving for!), or the game ends (with probability ε_5), and the payoff vector is $\left(\frac{b_3}{2}, \frac{b_3}{2}, 0\right) = \frac{b_3}{2}\delta_3$.

Adding together the products of the probabilities of the four events above and their expected payoffs, we find that:

$$\begin{aligned} UB2R1 \quad = \quad & (a_{21})(1-af_{21})\,UB3R21 + (1-a_{21})(af_{21})\,UB21R3 \\ & + (a_{21})(af_{21})\,\frac{1}{2}(UB21R3 + UB3R21) \\ & + (1-a_{21})(1-af_{21})\left((1-\varepsilon_5)\,UB2R1 + \varepsilon_5\frac{b_3}{2}\delta_3\right). \end{aligned}$$

Since $UB2R1$ appears on both sides we solve for it and simplify:

$$UB2R1 = \frac{\begin{array}{c}(a_{21})(2-af_{21})\,UB3R21 + (2-a_{21})(af_{21})\,UB21R3 \\ +(1-a_{21})(1-af_{21})\,\varepsilon_5 b_3\delta_3\end{array}}{2-2(1-a_{21})(1-af_{21})(1-\varepsilon_5)}. \tag{8}$$

Replacing 2 with i, 1 with j, and 3 with k, we can get the general form for this payoff vector:

$$UBiRj = \frac{\begin{array}{c}(a_{ij})(2-af_{ij})\,UBkRij + (2-a_{ij})(af_{ij})\,UBijRk \\ +(1-a_{ij})(1-af_{ij})\,\varepsilon_5 b_k\delta_k\end{array}}{2-2(1-a_{ij})(1-af_{ij})(1-\varepsilon_5)}. \tag{9}$$

Knowing this, we can now write down a formula for the total expected payoff vector for the entire game, U. Each player may either choose not to be represented or appoint one of two others to represent him, a total of three options. And since there are three players, each with three options, there are 27 possible combinations to consider. We can break them down to the following four combinations:

1. All three players choose another to represent them. Say for instance that $P1$ chooses $P2$, $P2$ chooses $P1$, and $P3$ also chooses $P1$. Then the combined probability of this event is $a_1f_2 \cdot a_2f_1 \cdot a_3f_1$. We flip a three-sided coin to determine which of the players will be represented. This brings us to round two, and we have already solved for the formulas there, so we write implicitly that the expected payoff vector is $\frac{1}{3}(UB2R1 + UB1R2 + UB1R3)$.

2. Two players choose to be represented while the third does not. An example of this type of occurrence is if $P1$ wants to go solo, but $P2$ accepts $P1$, and $P3$ accepts $P2$. The probability is then $n_1 \cdot a_2f_1 \cdot a_3f_2$, and as before the expected payoff (with outcome determined by a two-sided coin-toss) is $\frac{1}{2}(UB1R2 + UB2R3)$.

3. Only one player accepts another as a representative, and the other two decline. Say players $P1$ and $P3$ choose not to be represented while $P2$ accepts $P1$. The probability is $n_1 \cdot a_2 f_1 \cdot n_3$ and the payoff is simply $UB1R2$.

4. No player accepts another. The probability of this event is $n_1 \cdot n_2 \cdot n_3$. With probability ε_5, the payoff vector is $(0, 0, 0)$. Alternatively, the game is allowed to repeat, with expected payoff $(1 - \varepsilon_5)U$. Here, as before, we have U on both sides of the equation and must later solve for it.

Now we can write down all 27 possible occurrences and their expected payoffs:

$$
\begin{aligned}
U \quad = \quad & n_1 \cdot a_2 f_1 \cdot a_3 f_1 \cdot \frac{1}{2}\left(UB1R2 + UB1R3\right) \\
& + n_1 \cdot a_2 f_1 \cdot a_3 f_2 \cdot \frac{1}{2}\left(UB1R2 + UB2R3\right) \\
& + n_1 \cdot a_2 f_1 \cdot n_3 \cdot UB1R2 + n_1 \cdot n_2 \cdot a_3 f_1 \cdot UB1R3 \\
& + n_1 \cdot n_2 \cdot a_3 f_2 \cdot UB2R3 + n_1 \cdot a_2 f_3 \cdot n_3 \cdot UB3R2 \\
& + n_1 \cdot a_2 f_3 \cdot a_3 f_1 \cdot \frac{1}{2}\left(UB3R2 + UB1R3\right) \\
& + n_1 \cdot a_2 f_3 \cdot a_3 f_2 \cdot \frac{1}{2}\left(UB3R2 + UB2R3\right) \\
& + a_1 f_2 \cdot a_2 f_1 \cdot a_3 f_1 \cdot \frac{1}{3}\left(UB2R1 + UB1R2 + UB1R3\right) \\
& + a_1 f_2 \cdot a_2 f_1 \cdot a_3 f_2 \cdot \frac{1}{3}\left(UB2R1 + UB1R2 + UB2R3\right) \\
& + a_1 f_2 \cdot a_2 f_1 \cdot n_3 \cdot \frac{1}{2}\left(UB2R1 + UB1R2\right) \\
& + a_1 f_2 \cdot n_2 \cdot a_3 f_1 \cdot \frac{1}{2}\left(UB2R1 + UB1R3\right) \\
& + a_1 f_2 \cdot n_2 \cdot a_3 f_2 \cdot \frac{1}{2}\left(UB2R1 + UB2R3\right) \\
& + a_1 f_2 \cdot n_2 \cdot n_3 \cdot UB2R1 + a_1 f_3 \cdot n_2 \cdot n_3 \cdot UB3R1 \\
& + a_1 f_2 \cdot a_2 f_3 \cdot a_3 f_1 \cdot \frac{1}{3}\left(UB2R1 + UB3R2 + UB1R3\right) \\
& + a_1 f_2 \cdot a_2 f_3 \cdot a_3 f_2 \cdot \frac{1}{3}\left(UB2R1 + UB3R2 + UB2R3\right) \\
& + a_1 f_2 \cdot a_2 f_3 \cdot n_3 \cdot \frac{1}{2}\left(UB2R1 + UB3R2\right) \\
& + a_1 f_3 \cdot a_2 f_1 \cdot a_3 f_1 \cdot \frac{1}{3}\left(UB3R1 + UB1R2 + UB1R3\right) \\
& + a_1 f_3 \cdot a_2 f_1 \cdot a_3 f_2 \cdot \frac{1}{3}\left(UB3R1 + UB1R2 + UB2R3\right) \\
& + a_1 f_3 \cdot a_2 f_1 \cdot n_3 \cdot \frac{1}{2}\left(UB3R1 + UB1R2\right) \\
& + a_1 f_3 \cdot n_2 \cdot a_3 f_1 \cdot \frac{1}{2}\left(UB3R1 + UB1R3\right) \\
& + a_1 f_3 \cdot n_2 \cdot a_3 f_2 \cdot \frac{1}{2}\left(UB3R1 + UB2R3\right) \\
& + a_1 f_3 \cdot a_2 f_3 \cdot a_3 f_1 \cdot \frac{1}{3}\left(UB3R1 + UB3R2 + UB1R3\right) \\
& + a_1 f_3 \cdot a_2 f_3 \cdot a_3 f_2 \cdot \frac{1}{3}\left(UB3R1 + UB3R2 + UB2R3\right) \\
& + a_1 f_3 \cdot a_2 f_3 \cdot n_3 \cdot \frac{1}{2}\left(UB3R1 + UB3R2\right) \\
& + n_1 \cdot n_2 \cdot n_3 \cdot \left((1 - \varepsilon_4) \cdot U + \varepsilon_4 \cdot 0\right).
\end{aligned} \tag{10}
$$

Since we have U on both sides of the equation, we must again solve for it explicitly on one side.

So we have that the numerator of U is everything as it appears above except the last line, and the denominator is $(1 - n_1 \cdot n_2 \cdot n_3 (1 - \varepsilon_4))$.

It will be useful to consider another way to derive the same formula: to find the probabilities, $p\{iRj\}$. Say that in the second stage, $P2$ leads $P1$ and $P3$ is on his own. The payoff vector corresponding to this is $UB2R1$, which we have already solved for in (9). The event of $P2$ representing $P1$ has probability $p\{2R1\}$, and can come about in one of five scenarios:

1. $P1$ is the only player that wants to be represented; the other two decline. This has probability $a_1 f_2 \cdot n_2 \cdot n_3$.
2. $P1$ accepts $P2$, $P2$ accepts either $P1$ or $P3$, and $P3$ does not accept. Since $P1$ must win the two-sided coin toss, the total probability is $\frac{1}{2}(a_1 f_2 \cdot a_2 f_1 \cdot n_3 + a_1 f_2 \cdot a_2 f_3 \cdot n_3)$. Remembering equation (1), we can simplify to: $\frac{1}{2} a_1 f_2 \cdot (1 - n_2) \cdot n_3$.
3. $P1$ accepts $P2$, $P3$ accepts either $P1$ or $P2$, and $P2$ does not accept anyone. This is symmetric with the previous case, with total probability $\frac{1}{2} a_1 f_2 \cdot (1 - n_3) \cdot n_2$.
4. All three want to be represented, so a three-sided coin toss determines the acceptance. This has probability $\frac{1}{3} a_1 f_2 (1 - n_2)(1 - n_3)$.
5. All three decline to be represented, but are given another chance at round one, during which $P1$ successfully accepts $P2$. The probability here is $n_1 \cdot n_2 \cdot n_3 \cdot (1 - \varepsilon_4) \cdot p\{2R1\}$.

Summing these together, we get the formula for $p\{2R1\}$:

$$\begin{aligned} p\{2R1\} &= a_1 f_2 \cdot n_2 \cdot n_3 + \frac{1}{2} a_1 f_2 \cdot (1 - n_2) \cdot n_3 + \frac{1}{2} a_1 f_2 \cdot (1 - n_3) \cdot n_2 \\ &\quad + \frac{1}{3} a_1 f_2 (1 - n_2)(1 - n_3) + n_1 \cdot n_2 \cdot n_3 \cdot (1 - \varepsilon_4) \cdot p\{2R1\}. \end{aligned}$$

Solving for $p\{2R1\}$, we get:

$$p\{2R1\} = \frac{a_1 f_2 (2 + n_2 + n_3 + 2n_2 n_3)}{6(1 - n_1 \cdot n_2 \cdot n_3 \cdot (1 - \varepsilon_4))}. \tag{11}$$

Replacing 2 by i, 1 by j, and 3 by k, we get the general formula:

$$p\{iRj\} = \frac{a_j f_i}{6} \cdot \frac{2 + n_i + n_k + 2n_i n_k}{1 - n_1 \cdot n_2 \cdot n_3 \cdot (1 - \varepsilon_4)}. \tag{12}$$

Now we can write the formula for U using only $p\{iRj\}$ and $UBiRj$, both of which we have solved for:

$$\begin{aligned} U &= p\{1R2\} \cdot UB1R2 + p\{1R3\} \cdot UB1R3 + p\{2R1\} \cdot UB2R1 \\ &\quad + p\{2R3\} \cdot UB2R3 + p\{3R1\} \cdot UB3R1 + p\{3R2\} \cdot UB3R2. \end{aligned} \tag{13}$$

3 The Main Model

For each entry, u_i, of the total payoff vector, U, we have to solve thirteen equilibrium equations. That's thirty nine strategic parameters in all. For instance, only player $P1$ is concerned about maximizing u_1, and his only strategic variables (the variables which he controls) are d_1, df_{23}, d_{12}, df_{32}, d_{13}, $u_2b_1r_{23}$, $u_3b_1r_{23}$, $u_2b_1r_{32}$, $u_3b_1r_{32}$, $u_2b_{12}r_3$, $u_3b_{12}r_3$, $u_2b_{13}r_2$, and $u_3b_{13}r_2$. The derivative of u_1 with respect to all of these variables must then be set to zero, and the simultaneous solution to all of these equations is exactly the model. (Note: $P1$ also controls $u_1b_1r_{23}$, $u_1b_1r_{32}$, and so on, but they are dependent on the variables above, since $u_1b_1r_{23} = 1 - u_2b_1r_{23} - u_3b_1r_{23}$, etc.)

By symmetry, we only need to do this for $P1$, and for the variables d_1, df_{23}, d_{12}, $u_2b_1r_{23}$, $u_3b_1r_{23}$, $u_2b_{12}r_3$, and $u_3b_{12}r_3$. The rest is gotten by permuting the indeces. For variables df_{23} and d_{12}, we will only need derivatives with af_{23} and a_{12}, respectively, since these are the only parameters reacting with the d variables.

3.1 Variable d_1

So let us first consider the following specific case for $P1$, the problem of maximizing u_1 w.r.t. d_1. The first entry of the vector equation (13) is:

$$\begin{aligned} u_1 &= p\{1R2\} \cdot u_1b_1r_2 + p\{1R3\} \cdot u_1b_1r_3 + p\{2R1\} \cdot u_1b_2r_1 \\ &\quad + p\{2R3\} \cdot u_1b_2r_3 + p\{3R1\} \cdot u_1b_3r_1 + p\{3R2\} \cdot u_1b_3r_2. \end{aligned} \tag{14}$$

We will need the following derivatives from equation (2):

$$\begin{aligned} \frac{\partial A_1F_2}{\partial d_1} &= -\frac{A_1F_2}{\varepsilon_3}, \\ \frac{\partial A_1F_3}{\partial d_1} &= -\frac{A_1F_3}{\varepsilon_3}, \end{aligned} \tag{15}$$

and from equation (3):

$$\begin{aligned} \frac{\partial a_1f_2}{\partial A_1F_2} &= \frac{1 + A_1F_2 + A_1F_3 - A_1F_2}{(1 + A_1F_2 + A_1F_3)^2} = \frac{1 + A_1F_3}{(1 + A_1F_2 + A_1F_3)^2}, \\ \frac{\partial a_1f_2}{\partial A_1F_3} &= \frac{-A_1F_2}{(1 + A_1F_2 + A_1F_3)^2}, \\ \frac{\partial a_1f_3}{\partial A_1F_2} &= \frac{-A_1F_3}{(1 + A_1F_2 + A_1F_3)^2}, \\ \frac{\partial a_1f_3}{\partial A_1F_3} &= \frac{1 + A_1F_2}{(1 + A_1F_2 + A_1F_3)^2}. \end{aligned}$$

From here, we get:

$$\begin{aligned} \frac{\partial a_1f_2}{\partial d_1} &= \frac{\partial a_1f_2}{\partial A_1F_2}\frac{\partial A_1F_2}{\partial d_1} + \frac{\partial a_1f_2}{\partial A_1F_3}\frac{\partial A_1F_3}{\partial d_1} \\ &= \frac{1 + A_1F_3}{(1 + A_1F_2 + A_1F_3)^2}\frac{-A_1F_2}{\varepsilon_3} + \frac{A_1F_2}{(1 + A_1F_2 + A_1F_3)^2}\frac{A_1F_3}{\varepsilon_3} \\ &= \frac{1}{\varepsilon_3}\left(\frac{-A_1F_2}{(1 + A_1F_2 + A_1F_3)^2}\right) = \frac{-1}{\varepsilon_3}\left(a_1f_2 \cdot n_1\right), \end{aligned}$$

and similarly

$$\begin{aligned}
\frac{\partial a_1 f_3}{\partial d_1} &= \frac{-1}{\varepsilon_3}\left(a_1 f_3 \cdot n_1\right), \\
\frac{\partial n_1}{\partial d_1} &= -\left(\frac{\partial a_1 f_2}{\partial d_1} + \frac{\partial a_1 f_3}{\partial d_1}\right) = \frac{1}{\varepsilon_3}\left(\left(1 - n_1\right) \cdot n_1\right).
\end{aligned}$$

Since $p\{2R1\}$ reacts with n_1, and $a_1 f_2$ (from (11)), we get:

$$\begin{aligned}
\frac{\partial p\{2R1\}}{\partial n_1} &= \frac{a_1 f_2 \cdot n_2 \cdot n_3 \cdot \left(1 - \varepsilon_4\right)\left(2 + n_2 + n_3 + 2n_2 n_3\right)}{6\left(1 - n_1 \cdot n_2 \cdot n_3 \cdot \left(1 - \varepsilon_4\right)\right)^2}, \\
\frac{\partial p\{2R1\}}{\partial a_1 f_2} &= \frac{2 + n_2 + n_3 + 2n_2 n_3}{6\left(1 - n_1 \cdot n_2 \cdot n_3 \cdot \left(1 - \varepsilon_4\right)\right)},
\end{aligned}$$

from which we get $p\{3R1\}$ by replacing every 2 by a 3 and vice-versa. Then so far we have:

$$\begin{aligned}
\frac{\partial p\{2R1\}}{\partial d_1} &= \frac{\partial p\{2R1\}}{\partial n_1}\frac{\partial n_1}{\partial d_1} + \frac{\partial p\{2R1\}}{\partial a_1 f_2}\frac{\partial a_1 f_2}{\partial d_1} \\
&= \frac{\left(a_1 f_2 \cdot n_1\right)\left(2 + n_2 + n_3 + 2n_2 n_3\right)\left(n_2 \cdot n_3\left(1 - \varepsilon_4\right) - 1\right)}{6 \cdot \varepsilon_3 \cdot \left(1 - n_1 \cdot n_2 \cdot n_3 \cdot \left(1 - \varepsilon_4\right)\right)^2},
\end{aligned}$$

and

$$\frac{\partial p\{3R1\}}{\partial d_1} = \frac{\left(a_1 f_3 \cdot n_1\right)\left(2 + n_2 + n_3 + 2n_2 n_3\right)\left(n_2 \cdot n_3\left(1 - \varepsilon_4\right) - 1\right)}{6 \cdot \varepsilon_3 \cdot \left(1 - n_1 \cdot n_2 \cdot n_3 \cdot \left(1 - \varepsilon_4\right)\right)^2}.$$

Other $p\{\cdot\}$ variables only react with n_1, so we get:

$$\begin{aligned}
\frac{\partial p\{1R2\}}{\partial n_1} &= \frac{a_2 f_1\left(1 + n_3\left(2 + n_2\left(1 - \varepsilon_4\right)\left(2 + n_3\right)\right)\right)}{6\left(1 - n_1 \cdot n_2 \cdot n_3 \cdot \left(1 - \varepsilon_4\right)\right)^2}, \\
\frac{\partial p\{1R2\}}{\partial d_1} &= \frac{\partial p\{1R2\}}{\partial n_1}\frac{\partial n_1}{\partial d_1} \\
&= \frac{a_2 f_1\left(1 + n_3\left(2 + n_2\left(1 - \varepsilon_4\right)\left(2 + n_3\right)\right)\right)\left(\left(1 - n_1\right) \cdot n_1\right)}{6 \cdot \varepsilon_3\left(1 - n_1 \cdot n_2 \cdot n_3 \cdot \left(1 - \varepsilon_4\right)\right)^2}, \\
\frac{\partial p\{1R3\}}{\partial d_1} &= \frac{a_3 f_1\left(1 + n_2\left(2 + n_3\left(1 - \varepsilon_4\right)\left(2 + n_2\right)\right)\right)\left(\left(1 - n_1\right) \cdot n_1\right)}{6 \cdot \varepsilon_3\left(1 - n_1 \cdot n_2 \cdot n_3 \cdot \left(1 - \varepsilon_4\right)\right)^2}, \\
\frac{\partial p\{2R3\}}{\partial d_1} &= \frac{a_3 f_2\left(1 + n_2\left(2 + n_3\left(1 - \varepsilon_4\right)\left(2 + n_2\right)\right)\right)\left(\left(1 - n_1\right) \cdot n_1\right)}{6 \cdot \varepsilon_3\left(1 - n_1 \cdot n_2 \cdot n_3 \cdot \left(1 - \varepsilon_4\right)\right)^2}, \\
\frac{\partial p\{3R2\}}{\partial d_1} &= \frac{a_2 f_3\left(1 + n_3\left(2 + n_2\left(1 - \varepsilon_4\right)\left(2 + n_3\right)\right)\right)\left(\left(1 - n_1\right) \cdot n_1\right)}{6 \cdot \varepsilon_3\left(1 - n_1 \cdot n_2 \cdot n_3 \cdot \left(1 - \varepsilon_4\right)\right)^2}.
\end{aligned}$$

Combining these derivatives with equation (14), we get:

$$\begin{aligned}
\frac{\partial u_1}{\partial d_1} &= \frac{\partial p\{1R2\}}{\partial d_1}\cdot u_1b_1r_2+\frac{\partial p\{1R3\}}{\partial d_1}\cdot u_1b_1r_3+\frac{\partial p\{2R1\}}{\partial d_1}\cdot u_1b_2r_1\\
&\quad+\frac{\partial p\{2R3\}}{\partial d_1}\cdot u_1b_2r_3+\frac{\partial p\{3R1\}}{\partial d_1}\cdot u_1b_3r_1+\frac{\partial p\{3R2\}}{\partial d_1}\cdot u_1b_3r_2\\
&= \frac{a_2f_1\left(1+n_3\left(2+n_2\left(1-\varepsilon_4\right)\left(2+n_3\right)\right)\right)\left(\left(1-n_1\right)\cdot n_1\right)}{6\cdot\varepsilon_3\left(1-n_1\cdot n_2\cdot n_3\cdot\left(1-\varepsilon_4\right)\right)^2}\cdot u_1b_1r_2\\
&\quad+\frac{a_3f_1\left(1+n_2\left(2+n_3\left(1-\varepsilon_4\right)\left(2+n_2\right)\right)\right)\left(\left(1-n_1\right)\cdot n_1\right)}{6\cdot\varepsilon_3\left(1-n_1\cdot n_2\cdot n_3\cdot\left(1-\varepsilon_4\right)\right)^2}\cdot u_1b_1r_3\\
&\quad+\frac{\left(a_1f_2\cdot n_1\right)\left(2+n_2+n_3+2n_2n_3\right)\left(n_2\cdot n_3\left(1-\varepsilon_4\right)-1\right)}{6\cdot\varepsilon_3\cdot\left(1-n_1\cdot n_2\cdot n_3\cdot\left(1-\varepsilon_4\right)\right)^2}\cdot u_1b_2r_1\\
&\quad+\frac{a_3f_2\left(1+n_2\left(2+n_3\left(1-\varepsilon_4\right)\left(2+n_2\right)\right)\right)\left(\left(1-n_1\right)\cdot n_1\right)}{6\cdot\varepsilon_3\left(1-n_1\cdot n_2\cdot n_3\cdot\left(1-\varepsilon_4\right)\right)^2}\cdot u_1b_2r_3\\
&\quad+\frac{\left(a_1f_3\cdot n_1\right)\left(2+n_2+n_3+2n_2n_3\right)\left(n_2\cdot n_3\left(1-\varepsilon_4\right)-1\right)}{6\cdot\varepsilon_3\cdot\left(1-n_1\cdot n_2\cdot n_3\cdot\left(1-\varepsilon_4\right)\right)^2}\cdot u_1b_3r_1\\
&\quad+\frac{a_2f_3\left(1+n_3\left(2+n_2\left(1-\varepsilon_4\right)\left(2+n_3\right)\right)\right)\left(\left(1-n_1\right)\cdot n_1\right)}{6\cdot\varepsilon_3\left(1-n_1\cdot n_2\cdot n_3\cdot\left(1-\varepsilon_4\right)\right)^2}\cdot u_1b_3r_2.
\end{aligned}$$

And factoring $\frac{n_1}{6\cdot\varepsilon_3(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}$ out of everything and setting equal to zero, we get:

$$\begin{aligned}
0 &= \left(1+n_3\left(2+n_2\left(1-\varepsilon_4\right)\left(2+n_3\right)\right)\right)\left(1-n_1\right)\left(a_2f_1\cdot u_1b_1r_2+a_2f_3\cdot u_1b_3r_2\right)\\
&\quad+\left(1+n_2\left(2+n_3\left(1-\varepsilon_4\right)\left(2+n_2\right)\right)\right)\left(1-n_1\right)\left(a_3f_1\cdot u_1b_1r_3+a_3f_2\cdot u_1b_2r_3\right)\\
&\quad+\left(2+n_2+n_3+2n_2n_3\right)\left(n_2\cdot n_3\left(1-\varepsilon_4\right)-1\right)\left(a_1f_2\cdot u_1b_2r_1+a_1f_3\cdot u_1b_3r_1\right).
\end{aligned} \tag{16}$$

3.2 Variable df_{23}

Now let us consider the problem of maximizing u_1 w.r.t. df_{23}. As mentioned earlier, $\frac{\partial u_1}{\partial df_{23}} = \frac{\partial u_1}{\partial af_{23}}\frac{\partial af_{23}}{\partial df_{23}}$, so it suffices to consider just the equation with af_{23} and factor out $\frac{\partial af_{23}}{\partial df_{23}} = \frac{-af_{23}}{\varepsilon_3}$. So for af_{23}, the reactions to consider are with $u_1b_2r_3$ and $u_3b_2r_3$, which in turn react with a_3f_1, a_3f_2, and n_3, and then all of the $p\{iRj\}$ variables.

Then we will have:

$$\begin{aligned}
\frac{\partial u_1}{\partial af_{23}} &= p\{2R3\}\cdot\frac{\partial u_1b_2r_3}{\partial af_{23}}\\
&\quad+\frac{\partial p\{1R2\}}{\partial af_{23}}\cdot u_1b_1r_2+\frac{\partial p\{1R3\}}{\partial af_{23}}\cdot u_1b_1r_3+\frac{\partial p\{2R1\}}{\partial af_{23}}\cdot u_1b_2r_1\\
&\quad+\frac{\partial p\{2R3\}}{\partial af_{23}}\cdot u_1b_2r_3+\frac{\partial p\{3R1\}}{\partial af_{23}}\cdot u_1b_3r_1+\frac{\partial p\{3R2\}}{\partial af_{23}}\cdot u_1b_3r_2.
\end{aligned}$$

But for each of the $p\{iRj\}$, $\frac{\partial p\{iRj\}}{\partial af_{23}} = \frac{\partial p\{iRj\}}{\partial u_3 b_2 r_3}\frac{\partial u_3 b_2 r_3}{\partial af_{23}}$, and so we calculate:

$$
\begin{aligned}
\frac{\partial p\{1R2\}}{\partial u_3 b_2 r_3} &= \frac{-a_2 f_1 (a_3 f_2)\, n_3 (1 + n_1 (2 + n_2 (1-\varepsilon_4)(n_1+2)))}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}\\
\frac{\partial p\{2R1\}}{\partial u_3 b_2 r_3} &= \frac{-a_1 f_2 (a_3 f_2)\, n_3 (1 + n_2 (2 + n_1 (1-\varepsilon_4)(n_2+2)))}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}\\
\frac{\partial p\{3R1\}}{\partial u_3 b_2 r_3} &= \frac{-a_1 f_3 (a_3 f_2)\, n_3 (1 + n_2 (2 + n_1 (1-\varepsilon_4)(n_2+2)))}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}\\
\frac{\partial p\{3R2\}}{\partial u_3 b_2 r_3} &= \frac{-a_2 f_3 (a_3 f_2)\, n_3 (1 + n_1 (2 + n_2 (1-\varepsilon_4)(n_1+2)))}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}\\
\frac{\partial p\{1R3\}}{\partial u_3 b_2 r_3} &= \frac{-a_3 f_1 (a_3 f_2)(2 + n_1 + n_2 + 2n_1 n_2)}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}\\
\frac{\partial p\{2R3\}}{\partial u_3 b_2 r_3} &= \frac{a_3 f_2 (2 + n_1 + n_2 + 2n_1 n_2)(1 - a_3 f_2 - n_1\cdot n_2\cdot n_3 (1-\varepsilon_4))}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}
\end{aligned}
$$

We also need:

$$
\begin{aligned}
\frac{\partial u_3 b_2 r_3}{\partial af_{23}} &= \frac{\begin{array}{c} u_3 b_1 r_{23} (a_{23} (\varepsilon_5 + a_{23}(1-\varepsilon_5) - 2)) \\ + u_3 b_{23} r_1 (2 - a_{23})(\varepsilon_5 + a_{23}(1-\varepsilon_5)) \\ -(1-a_{23})\cdot b_1\cdot\varepsilon_5 \end{array}}{2(1-(1-af_{23})(1-a_{23})(1-\varepsilon_5))^2},\\
\frac{\partial u_1 b_2 r_3}{\partial af_{23}} &= \frac{\begin{array}{c} u_1 b_1 r_{23} (a_{23} (\varepsilon_5 + a_{23}(1-\varepsilon_5) - 2)) \\ + u_1 b_{23} r_1 (2 - a_{23})(\varepsilon_5 + a_{23}(1-\varepsilon_5)) \end{array}}{2(1-(1-af_{23})(1-a_{23})(1-\varepsilon_5))^2},
\end{aligned}
$$

for the final form (pulling out $\frac{a_3 f_2}{6\varepsilon_3(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2\cdot 2(1-(1-af_{23})(1-a_{23})(1-\varepsilon_5))^2}$):

$$
\begin{aligned}
0 = {} & \varepsilon_3 (2 + n_2 + n_1 + 2n_1\cdot n_2)(1 - n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))\\
& \cdot (u_1 b_1 r_{23} (a_{23}(\varepsilon_5 + a_{23}(1-\varepsilon_5) - 2)) + u_1 b_{23} r_1 (2 - a_{23})(\varepsilon_5 + a_{23}(1-\varepsilon_5)))\\
& - \begin{pmatrix} u_3 b_1 r_{23} (a_{23}(\varepsilon_5 + a_{23}(1-\varepsilon_5) - 2)) \\ + u_3 b_{23} r_1 (2 - a_{23})(\varepsilon_5 + a_{23}(1-\varepsilon_5)) - (1-a_{23})\cdot b_1\cdot\varepsilon_5 \end{pmatrix}\\
& \cdot \begin{pmatrix} n_3 \begin{pmatrix} (1 + n_1 (2 + n_2 (1-\varepsilon_4)(n_1+2)))(a_2 f_1\cdot u_1 b_1 r_2 + a_2 f_3\cdot u_1 b_3 r_2) \\ + (1 + n_2 (2 + n_1 (1-\varepsilon_4)(n_2+2)))(a_1 f_2\cdot u_1 b_2 r_1 + a_1 f_3\cdot u_1 b_3 r_1) \end{pmatrix} \\ + (2 + n_1 + n_2 + 2n_1 n_2)(a_3 f_1\cdot u_1 b_1 r_3 - (1 - a_3 f_2 - n_1\cdot n_2\cdot n_3 (1-\varepsilon_4))\cdot u_1 b_2 r_3) \end{pmatrix}.
\end{aligned}
$$

3.3 Variable d_{12}

Now let us consider the problem of maximizing u_1 w.r.t. d_{12}. Again it will suffice to consider just the equation with a_{12} and factor out $\frac{\partial a_{12}}{\partial d_{12}} = \frac{-a_{12}}{\varepsilon_3}$. So for a_{12}, the reactions to consider are with $u_1 b_1 r_2$ and $u_2 b_1 r_2$, which in turn react with $a_2 f_1$, $a_2 f_3$, and n_2, and then all of the $p\{iRj\}$ variables.

Then we will have:

$$\begin{aligned}\frac{\partial u_1}{\partial a_{12}} &= p\{1R2\}\cdot\frac{\partial u_1 b_1 r_2}{\partial a_{12}}\\ &\quad+\frac{\partial p\{1R2\}}{\partial a_{12}}\cdot u_1 b_1 r_2+\frac{\partial p\{1R3\}}{\partial a_{12}}\cdot u_1 b_1 r_3+\frac{\partial p\{2R1\}}{\partial a_{12}}\cdot u_1 b_2 r_1\\ &\quad+\frac{\partial p\{2R3\}}{\partial a_{12}}\cdot u_1 b_2 r_3+\frac{\partial p\{3R1\}}{\partial a_{12}}\cdot u_1 b_3 r_1+\frac{\partial p\{3R2\}}{\partial a_{12}}\cdot u_1 b_3 r_2.\end{aligned}$$

But for each of the $p\{iRj\}$, $\frac{\partial p\{iRj\}}{\partial a_{12}}=\frac{\partial p\{iRj\}}{\partial u_2 b_1 r_2}\frac{\partial u_2 b_1 r_2}{\partial a_{12}}$, and so we calculate:

$$\begin{aligned}\frac{\partial p\{1R2\}}{\partial u_2 b_1 r_2} &= \frac{a_2 f_1(2+n_1+n_3+2n_1 n_3)(1-a_2 f_1-n_1\cdot n_2\cdot n_3(1-\varepsilon_4))}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2},\\ \frac{\partial p\{3R2\}}{\partial u_2 b_1 r_2} &= \frac{-a_2 f_1(a_2 f_3)(2+n_1+n_3+2n_1 n_3)}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2},\\ \frac{\partial p\{1R3\}}{\partial u_2 b_1 r_2} &= \frac{-a_2 f_1(a_3 f_1)n_2(1+n_1(2+n_3(1-\varepsilon_4)(n_1+2)))}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}\\ \frac{\partial p\{2R1\}}{\partial u_2 b_1 r_2} &= \frac{-a_2 f_1(a_1 f_2)n_2(1+n_3(2+n_1(1-\varepsilon_4)(n_3+2)))}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}\\ \frac{\partial p\{2R3\}}{\partial u_2 b_1 r_2} &= \frac{-a_2 f_1(a_3 f_2)n_2(1+n_1(2+n_3(1-\varepsilon_4)(n_1+2)))}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}\\ \frac{\partial p\{3R1\}}{\partial u_2 b_1 r_2} &= \frac{-a_2 f_1(a_1 f_3)n_2(1+n_3(2+n_1(1-\varepsilon_4)(n_3+2)))}{6\cdot\varepsilon_3\cdot(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}\end{aligned}$$

We also need:

$$\begin{aligned}\frac{\partial u_1 b_1 r_2}{\partial a_{12}} &= \frac{\begin{array}{c}u_1 b_{12} r_3(af_{12})(\varepsilon_5+af_{12}(1-\varepsilon_5)-2)\\ +u_1 b_3 r_{12}(2-af_{12})(\varepsilon_5+af_{12}(1-\varepsilon_5))\\ +(1-af_{12})\cdot b_3\cdot\varepsilon_5\end{array}}{2(1-(1-a_{12})(1-af_{12})(1-\varepsilon_5))^2},\\ \frac{\partial u_2 b_1 r_2}{\partial a_{12}} &= \frac{\begin{array}{c}u_2 b_{12} r_3(af_{12})(\varepsilon_5+af_{12}(1-\varepsilon_5)-2)\\ +u_2 b_3 r_{12}(2-af_{12})(\varepsilon_5+af_{12}(1-\varepsilon_5))\\ +(1-af_{12})\cdot b_3\cdot\varepsilon_5\end{array}}{2(1-(1-a_{12})(1-af_{12})(1-\varepsilon_5))^2},\end{aligned}$$

for the final form (pulling out $\frac{a_2 f_1}{6\varepsilon_3(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2\cdot 2(1-(1-a_{12})(1-af_{12})(1-\varepsilon_5))^2}$):

$$\begin{aligned}0 &= \varepsilon_3(2+n_3+n_1+2n_1\cdot n_3)(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))\\ &\quad\cdot\left(\begin{array}{c}u_1 b_{12} r_3(af_{12})(\varepsilon_5+af_{12}(1-\varepsilon_5)-2)\\ +u_1 b_3 r_{12}(2-af_{12})(\varepsilon_5+af_{12}(1-\varepsilon_5))+(1-af_{12})\cdot b_3\cdot\varepsilon_5\end{array}\right)\\ &\quad-\left(\begin{array}{c}u_2 b_{12} r_3(af_{12})(\varepsilon_5+af_{12}(1-\varepsilon_5)-2)\\ +u_2 b_3 r_{12}(2-af_{12})(\varepsilon_5+af_{12}(1-\varepsilon_5))+(1-af_{12})\cdot b_3\cdot\varepsilon_5\end{array}\right)\\ &\quad\cdot\left(\begin{array}{c}(2+n_1+n_3+2n_1 n_3)((a_2 f_1+n_1\cdot n_2\cdot n_3(1-\varepsilon_4)-1)\cdot u_1 b_1 r_2+a_2 f_3\cdot u_1 b_3 r_2)\\ +n_2\left(\begin{array}{c}(1+n_1(2+n_3(1-\varepsilon_4)(n_1+2)))(a_3 f_1\cdot u_1 b_1 r_3+a_3 f_2\cdot u_1 b_2 r_3)\\ +(1+n_3(2+n_1(1-\varepsilon_4)(n_3+2)))(a_1 f_2\cdot u_1 b_2 r_1+a_1 f_3\cdot u_1 b_3 r_1)\end{array}\right)\end{array}\right).\end{aligned}$$

3.4 Variable $u_2b_{12}r_3$

The next equation we will consider is the derivative of u_1 with respect to $u_2b_{12}r_3$, since this is a strategic parameter controlled by $P1$. For this, we will need to calculate the corresponding partial derivatives. The only new reactions needed are those with n_2, a_2f_3, a_2f_1, $u_2b_1r_2$, and $u_1b_1r_2$, and we will use formulas derived in the previous section for the full derivative. In other words, once we know $\frac{\partial a_2f_1}{\partial u_2b_{12}r_3} = \frac{\partial a_2f_1}{\partial u_2b_1r_2} \cdot \frac{\partial u_2b_1r_2}{\partial u_2b_{12}r_3}, \frac{\partial a_2f_3}{\partial u_2b_{12}r_3}, \frac{\partial n_2}{\partial u_2b_{12}r_3}$, and $\frac{\partial u_1b_1r_2}{\partial u_2b_{12}r_3}$, we will simply have $\frac{\partial p_if_j}{\partial u_2b_{12}r_3} = \frac{\partial p_if_j}{\partial a_jf_i} \cdot \frac{\partial a_jf_i}{\partial u_2b_{12}r_3} + \frac{\partial p_if_j}{\partial n_2} \cdot \frac{\partial n_2}{\partial u_2b_{12}r_3}$, and:

$$\begin{aligned}\frac{\partial u_1}{\partial u_2b_{12}r_3} &= p\{1R2\} \cdot \frac{\partial u_1b_1r_2}{\partial u_2b_{12}r_3} \\ &\quad + \frac{\partial p\{1R2\}}{\partial u_2b_{12}r_3} \cdot u_1b_1r_2 + \frac{\partial p\{1R3\}}{\partial u_2b_{12}r_3} \cdot u_1b_1r_3 + \frac{\partial p\{2R1\}}{\partial u_2b_{12}r_3} \cdot u_1b_2r_1 \\ &\quad + \frac{\partial p\{2R3\}}{\partial u_2b_{12}r_3} \cdot u_1b_2r_3 + \frac{\partial p\{3R1\}}{\partial u_2b_{12}r_3} \cdot u_1b_3r_1 + \frac{\partial p\{3R2\}}{\partial u_2b_{12}r_3} \cdot u_1b_3r_2.\end{aligned} \tag{17}$$

We will need the following derivatives from equation:

$$\frac{\partial A_2F_1}{\partial u_2b_1r_2} = \frac{A_2F_1}{\varepsilon_3},$$

and

$$\begin{aligned}\frac{\partial a_2f_1}{\partial A_2F_1} &= \frac{1 + A_2F_3}{(1 + A_2F_1 + A_2F_3)^2}, \\ \frac{\partial a_2f_3}{\partial A_2F_1} &= \frac{-A_2F_3}{(1 + A_2F_1 + A_2F_3)^2}.\end{aligned}$$

From here, we get:

$$\begin{aligned}\frac{\partial a_2f_1}{\partial u_2b_1r_2} &= \frac{\partial a_2f_1}{\partial A_2F_1}\frac{\partial A_2F_1}{\partial u_2b_1r_2} \\ &= \frac{1 + A_2F_3}{(1 + A_2F_1 + A_2F_3)^2}\frac{A_2F_1}{\varepsilon_3} \\ &= \frac{a_2f_1(1 - a_2f_1)}{\varepsilon_3},\end{aligned}$$

and similarly

$$\begin{aligned}\frac{\partial a_2f_3}{\partial u_2b_1r_2} &= \frac{\partial a_2f_3}{\partial A_2F_1}\frac{\partial A_2F_1}{\partial u_2b_1r_2} = -\frac{a_2f_3 \cdot a_2f_1}{\varepsilon_3}, \\ \frac{\partial n_2}{\partial u_2b_1r_2} &= -\left(\frac{\partial a_2f_1}{\partial u_2b_1r_2} + \frac{\partial a_2f_3}{\partial u_2b_1r_2}\right) = -\frac{a_2f_1(n_2)}{\varepsilon_3}.\end{aligned}$$

Then as before, we have:

$$\begin{aligned}
\frac{\partial p\{1R2\}}{\partial u_2 b_1 r_2} &= \frac{\partial p\{1R2\}}{\partial n_2}\frac{\partial n_2}{\partial u_2 b_1 r_2} + \frac{\partial p\{1R2\}}{\partial a_2 f_1}\frac{\partial a_2 f_1}{\partial u_2 b_1 r_2} \\
&= \frac{a_2 f_1 (2 + n_1 + n_3 + 2n_1 n_3)(1 - a_2 f_1 - n_1 \cdot n_2 \cdot n_3 (1-\varepsilon_4))}{6 \cdot \varepsilon_3 \cdot (1 - n_1 \cdot n_2 \cdot n_3 \cdot (1-\varepsilon_4))^2}, \\
\frac{\partial p\{3R2\}}{\partial u_2 b_1 r_2} &= \frac{-a_2 f_1 (a_2 f_3)(2 + n_1 + n_3 + 2n_1 n_3)}{6 \cdot \varepsilon_3 \cdot (1 - n_1 \cdot n_2 \cdot n_3 \cdot (1-\varepsilon_4))^2}, \\
\frac{\partial p\{1R3\}}{\partial u_2 b_1 r_2} &= \frac{-a_2 f_1 (a_3 f_1) n_2 (1 + n_1 (2 + n_3 (1-\varepsilon_4)(n_1+2)))}{6 \cdot \varepsilon_3 \cdot (1 - n_1 \cdot n_2 \cdot n_3 \cdot (1-\varepsilon_4))^2} \\
\frac{\partial p\{2R1\}}{\partial u_2 b_1 r_2} &= \frac{-a_2 f_1 (a_1 f_2) n_2 (1 + n_3 (2 + n_1 (1-\varepsilon_4)(n_3+2)))}{6 \cdot \varepsilon_3 \cdot (1 - n_1 \cdot n_2 \cdot n_3 \cdot (1-\varepsilon_4))^2} \\
\frac{\partial p\{2R3\}}{\partial u_2 b_1 r_2} &= \frac{-a_2 f_1 (a_3 f_2) n_2 (1 + n_1 (2 + n_3 (1-\varepsilon_4)(n_1+2)))}{6 \cdot \varepsilon_3 \cdot (1 - n_1 \cdot n_2 \cdot n_3 \cdot (1-\varepsilon_4))^2} \\
\frac{\partial p\{3R1\}}{\partial u_2 b_1 r_2} &= \frac{-a_2 f_1 (a_1 f_3) n_2 (1 + n_3 (2 + n_1 (1-\varepsilon_4)(n_3+2)))}{6 \cdot \varepsilon_3 \cdot (1 - n_1 \cdot n_2 \cdot n_3 \cdot (1-\varepsilon_4))^2}
\end{aligned}$$

Combining these derivatives with equation (17), and pulling out a factor of $\frac{-a_2 f_1}{6\varepsilon_3(1-(1-\varepsilon_4)n_1 \cdot n_2 \cdot n_3)^2}\frac{(2-a_{12})(af_{12})}{2(1-(1-a_{12})(1-af_{12})(1-\varepsilon_5))}$, we get:

$$\begin{aligned}
0 &= \varepsilon_3 (2 + n_1 + n_3 + 2n_1 \cdot n_3)(1 - (1-\varepsilon_4) n_1 \cdot n_2 \cdot n_3) \\
&\quad + (2 + n_1 + n_3 + 2n_1 n_3)((a_2 f_1 + n_1 \cdot n_2 \cdot n_3 (1-\varepsilon_4) - 1) \cdot u_1 b_1 r_2 + a_2 f_3 \cdot u_1 b_3 r_2) \\
&\quad + n_2 \begin{pmatrix} (1 + n_1 (2 + n_3 (1-\varepsilon_4)(n_1+2)))(a_3 f_1 \cdot u_1 b_1 r_3 + a_3 f_2 \cdot u_1 b_2 r_3) \\ + (1 + n_3 (2 + n_1 (1-\varepsilon_4)(n_3+2)))(a_1 f_2 \cdot u_1 b_2 r_1 + a_1 f_3 \cdot u_1 b_3 r_1) \end{pmatrix}.
\end{aligned}$$

3.5 Variable $u_3b_{12}r_3$

The next equation we will consider is the derivative of u_1 with respect to $u_3b_{12}r_3$. The lower level reactions occur with af_{12}, $u_2b_1r_2$ and $u_1b_1r_2$. So, we find:

$$\begin{aligned}
\frac{\partial af_{12}}{\partial u_3b_{12}r_3} &= \frac{af_{12}\left(1-af_{12}\right)}{\varepsilon_3}, \\
\frac{\partial u_1b_1r_2}{\partial u_3b_{12}r_3} &= \frac{af12}{2\varepsilon_3\left(1-\left(1-a_{12}\right)\left(1-af_{12}\right)\left(1-\varepsilon_5\right)\right)^2} \\
&\cdot\left(\begin{array}{c} \varepsilon_3\left(a_{12}-2\right)\left(1-\left(1-a_{12}\right)\left(1-af_{12}\right)\left(1-\varepsilon_5\right)\right) \\ +\left(1-af_{12}\right)\left(\begin{array}{c}\left(u_1b_{12}r_3\right)\left(2-a_{12}\right)\left(\varepsilon_5+a_{12}\left(1-\varepsilon_5\right)\right) \\ +u_1b_3r_{12}\left(a_{12}\right)\left(\varepsilon_5+a_{12}\left(1-\varepsilon_5\right)-2\right)+\left(a_{12}-1\right)\cdot b_3\cdot\varepsilon_5\end{array}\right)\end{array}\right) \\
\frac{\partial u_2b_1r_2}{\partial af_{12}} &= \frac{\left(\begin{array}{c}\left(u_2b_{12}r_3\right)\left(2-a_{12}\right)\left(\varepsilon_5+a_{12}\left(1-\varepsilon_5\right)\right) \\ +u_2b_3r_{12}\left(a_{12}\right)\left(\varepsilon_5+a_{12}\left(1-\varepsilon_5\right)-2\right)+\left(a_{12}-1\right)\cdot b_3\cdot\varepsilon_5\end{array}\right)}{2\left(1-\left(1-a_{12}\right)\left(1-af_{12}\right)\left(1-\varepsilon_5\right)\right)^2} \\
\frac{\partial u_2b_1r_2}{\partial u_3b_{12}r_3} &= \frac{\partial u_2b_1r_2}{\partial af_{12}}\frac{\partial af_{12}}{\partial u_3b_{12}r_3}.
\end{aligned}$$

Then we have the full form:

$$\begin{aligned}
\frac{\partial u_1}{\partial u_3b_{12}r_3} &= p\left\{1R2\right\}\cdot\frac{\partial u_1b_1r_2}{\partial u_3b_{12}r_3} \\
&+\frac{\partial u_2b_1r_2}{\partial u_3b_{12}r_3}\left(\begin{array}{c}\frac{\partial p\{1R2\}}{\partial u_2b_1r_2}\cdot u_1b_1r_2+\frac{\partial p\{1R3\}}{\partial u_2b_1r_2}\cdot u_1b_1r_3 \\ +\frac{\partial p\{2R1\}}{\partial u_2b_1r_2}\cdot u_1b_2r_1+\frac{\partial p\{2R3\}}{\partial u_2b_1r_2}\cdot u_1b_2r_3 \\ +\frac{\partial p\{3R1\}}{\partial u_2b_1r_2}\cdot u_1b_3r_1+\frac{\partial p\{3R2\}}{\partial u_2b_1r_2}\cdot u_1b_3r_2\end{array}\right),
\end{aligned}$$

and pulling out $\frac{a_2f_1}{6\varepsilon_3(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}\frac{af_{12}}{2\varepsilon_3(1-(1-a_{12})(1-af_{12})(1-\varepsilon_5))^2}$, we get:

$$\begin{aligned}
0 &= \varepsilon_3\left(2+n_1+n_3+2n_1\cdot n_3\right)\left(1-n_1\cdot n_2\cdot n_3\cdot\left(1-\varepsilon_4\right)\right) \\
&\cdot\left(\begin{array}{c}\varepsilon_3\left(a_{12}-2\right)\left(1-\left(1-a_{12}\right)\left(1-af_{12}\right)\left(1-\varepsilon_5\right)\right) \\ +\left(1-af_{12}\right)\left(\begin{array}{c}\left(u_1b_{12}r_3\right)\left(2-a_{12}\right)\left(\varepsilon_5+a_{12}\left(1-\varepsilon_5\right)\right) \\ +u_1b_3r_{12}\left(a_{12}\right)\left(\varepsilon_5+a_{12}\left(1-\varepsilon_5\right)-2\right)+\left(a_{12}-1\right)\cdot b_3\cdot\varepsilon_5\end{array}\right)\end{array}\right) \\
&-\left(\begin{array}{c}\left(u_2b_{12}r_3\right)\left(2-a_{12}\right)\left(\varepsilon_5+a_{12}\left(1-\varepsilon_5\right)\right) \\ +u_2b_3r_{12}\left(a_{12}\right)\left(\varepsilon_5+a_{12}\left(1-\varepsilon_5\right)-2\right)+\left(a_{12}-1\right)\cdot b_3\cdot\varepsilon_5\end{array}\right)\left(1-af_{12}\right) \\
&\cdot\left(\begin{array}{c}\left(2+n_1+n_3+2n_1n_3\right)\left(\left(a_2f_1+n_1\cdot n_2\cdot n_3\left(1-\varepsilon_4\right)-1\right)\cdot u_1b_1r_2+a_2f_3\cdot u_1b_3r_2\right) \\ +n_2\left(\begin{array}{c}\left(a_3f_1\cdot u_1b_1r_3+a_3f_2\cdot u_1b_2r_3\right)\left(1+n_1\left(2+n_3\left(1-\varepsilon_4\right)\left(n_1+2\right)\right)\right) \\ +\left(a_1f_2\cdot u_1b_2r_1+a_1f_3\cdot u_1b_3r_1\right)\left(1+n_3\left(2+n_1\left(1-\varepsilon_4\right)\left(n_3+2\right)\right)\right)\end{array}\right)\end{array}\right),
\end{aligned}$$

3.6 Variable $u_2b_1r_{23}$

The next equation we will consider is the derivative of u_1 with respect to $u_2b_1r_{23}$. The lower level reactions occur with a_{23}, a_3f_1, a_3f_2, n_3, $u_2b_2r_3$ and $u_1b_2r_3$. So, we find:

$$\begin{aligned}
\frac{\partial a_{23}}{\partial u_2 b_1 r_{23}} &= \frac{a_{23}\left(1-a_{23}\right)}{\varepsilon_3}, \\
\frac{\partial a_3 f_1}{\partial u_3 b_2 r_3} &= -\frac{a_3 f_1 \cdot a_3 f_2}{\varepsilon_3}, \\
\frac{\partial a_3 f_2}{\partial u_3 b_2 r_3} &= \frac{a_3 f_2\left(1-a_3 f_2\right)}{\varepsilon_3} \\
\frac{\partial u_1 b_2 r_3}{\partial u_2 b_1 r_{23}} &= -\frac{\partial u_1 b_2 r_3}{\partial u_1 b_1 r_{23}} + \frac{\partial u_1 b_2 r_3}{\partial a_{23}} \frac{\partial a_{23}}{\partial u_2 b_1 r_{23}} \\
&= a_{23} \frac{\left(1-a_{23}\right)\left(\begin{array}{c}\left(u_1 b_1 r_{23}\right)\left(2-a f_{23}\right)\left(\varepsilon_5+a f_{23}\left(1-\varepsilon_5\right)\right) \\ +\left(u_1 b_{23} r_1\right) a f_{23}\left(\varepsilon_5+a f_{23}\left(1-\varepsilon_5\right)-2\right)\end{array}\right) - \varepsilon_3\left(2-a f_{23}\right)\left(1-\left(1-a f_{23}\right)\left(1-a_{23}\right)\left(1-\varepsilon_5\right)\right)}{2\left(1-\left(1-a f_{23}\right)\left(1-a_{23}\right)\left(1-\varepsilon_5\right)\right)^2 \varepsilon_3},
\end{aligned}$$

$$\frac{\partial u_3 b_2 r_3}{\partial u_2 b_1 r_{23}} = a_{23} \frac{\left(1-a_{23}\right)\left(\begin{array}{c} u_3 b_1 r_{23}\left(2-a f_{23}\right)\left(\varepsilon_5+a f_{23}\left(1-\varepsilon_5\right)\right) \\ +u_3 b_{23} r_1\left(a f_{23}\right)\left(\varepsilon_5+a f_{23}\left(1-\varepsilon_5\right)-2\right)-b_1 \cdot \varepsilon_5\left(1-a f_{23}\right)\end{array}\right)}{2\left(1-\left(1-a f_{23}\right)\left(1-a_{23}\right)\left(1-\varepsilon_5\right)\right)^2 \varepsilon_3}$$

Then we pull out $\frac{a_3 f_2}{6\varepsilon_3\left(1-n_1 \cdot n_2 \cdot n_3 \cdot\left(1-\varepsilon_4\right)\right)^2} \frac{a_{23}}{2\left(1-\left(1-a f_{23}\right)\left(1-a_{23}\right)\left(1-\varepsilon_5\right)\right)^2 \varepsilon_3}$, we get:

$$\begin{aligned}
0 = {} & \varepsilon_3\left(2+n_1+n_2+2 n_1 \cdot n_2\right)\left(1-n_1 \cdot n_2 \cdot n_3 \cdot\left(1-\varepsilon_4\right)\right) \\
& \cdot\left(\left(1-a_{23}\right)\left(\begin{array}{c}\left(u_1 b_1 r_{23}\right)\left(2-a f_{23}\right)\left(\varepsilon_5+a f_{23}\left(1-\varepsilon_5\right)\right) \\ +\left(u_1 b_{23} r_1\right) a f_{23}\left(\varepsilon_5+a f_{23}\left(1-\varepsilon_5\right)-2\right)\end{array}\right)-\varepsilon_3\left(2-a f_{23}\right)\left(1-\left(1-a f_{23}\right)\left(1-a_{23}\right)\left(1-\varepsilon_5\right)\right)\right) \\
& -\left(\left(1-a_{23}\right)\left(\begin{array}{c} u_3 b_1 r_{23}\left(2-a f_{23}\right)\left(\varepsilon_5+a f_{23}\left(1-\varepsilon_5\right)\right) \\ +u_3 b_{23} r_1\left(a f_{23}\right)\left(\varepsilon_5+a f_{23}\left(1-\varepsilon_5\right)-2\right)-b_1 \cdot \varepsilon_5\left(1-a f_{23}\right)\end{array}\right)\right) \\
& \cdot\left(\begin{array}{c} n_3\left(\begin{array}{c}\left(1+n_1\left(2+n_2\left(1-\varepsilon_4\right)\left(n_1+2\right)\right)\right)\left(a_2 f_1 \cdot u_1 b_1 r_2+a_2 f_3 \cdot u_1 b_3 r_2\right) \\ +\left(1+n_2\left(2+n_1\left(1-\varepsilon_4\right)\left(n_2+2\right)\right)\right)\left(a_1 f_2 \cdot u_1 b_2 r_1+a_1 f_3 \cdot u_1 b_3 r_1\right)\end{array}\right) \\ +\left(2+n_1+n_2+2 n_1 n_2\right)\left(a_3 f_1 \cdot u_1 b_1 r_3-\left(1-a_3 f_2-n_1 \cdot n_2 \cdot n_3\left(1-\varepsilon_4\right)\right) \cdot u_1 b_2 r_3\right)\end{array}\right)
\end{aligned}$$

3.7 Variable $u_3b_1r_{23}$

The last equation we will consider is the derivative of u_1 with respect to $u_3b_1r_{23}$. The lower level reactions occur with a_3f_1, a_3f_2, n_3, $u_3b_2r_3$ and $u_1b_2r_3$. So, we find:

$$\begin{aligned}
\frac{\partial a_3f_1}{\partial u_3b_2r_3} &= -\frac{a_3f_1 \cdot a_3f_2}{\varepsilon_3}, \\
\frac{\partial a_3f_2}{\partial u_3b_2r_3} &= \frac{a_3f_2\left(1-a_3f_2\right)}{\varepsilon_3} \\
\frac{\partial u_1b_2r_3}{\partial u_3b_1r_{23}} &= -\frac{\partial u_1b_2r_3}{\partial u_1b_1r_{23}} = \frac{-a_{23}\left(2-af_{23}\right)}{2\left(1-\left(1-af_{23}\right)\left(1-a_{23}\right)\left(1-\varepsilon_5\right)\right)}, \\
\frac{\partial u_3b_2r_3}{\partial u_3b_1r_{23}} &= a_{23}\frac{\left(2-af_{23}\right)}{2\left(1-\left(1-af_{23}\right)\left(1-a_{23}\right)\left(1-\varepsilon_5\right)\right)}
\end{aligned}$$

Then we pull out $\frac{-a_3f_2}{6\varepsilon_3(1-n_1\cdot n_2\cdot n_3\cdot(1-\varepsilon_4))^2}\frac{a_{23}(2-af_{23})}{2(1-(1-af_{23})(1-a_{23})(1-\varepsilon_5))}$, we get:

$$\begin{aligned}
0 \;=\; & \varepsilon_3\left(2+n_1+n_2+2n_1\cdot n_2\right)\left(1-n_1\cdot n_2\cdot n_3\cdot\left(1-\varepsilon_4\right)\right) \\
& + n_3\begin{pmatrix} \left(1+n_1\left(2+n_2\left(1-\varepsilon_4\right)\left(n_1+2\right)\right)\right)\left(a_2f_1\cdot u_1b_1r_2+a_2f_3\cdot u_1b_3r_2\right) \\ +\left(1+n_2\left(2+n_1\left(1-\varepsilon_4\right)\left(n_2+2\right)\right)\right)\left(a_1f_2\cdot u_1b_2r_1+a_1f_3\cdot u_1b_3r_1\right)\end{pmatrix} \\
& +\left(2+n_1+n_2+2n_1n_2\right)\left(a_3f_1\cdot u_1b_1r_3-\left(1-a_3f_2-n_1\cdot n_2\cdot n_3\left(1-\varepsilon_4\right)\right)\cdot u_1b_2r_3\right).
\end{aligned}$$