#1 Recall that if $A$ is an $m \times n$ matrix with entries from a field $\mathbb{F}$ then the linear transformation $L_A : \mathbb{F}^n \to \mathbb{F}^m$ is defined by $L_A(x) = Ax$.

(a) Prove that the system of linear equations $Ax = b$ has a solution $\iff b \in \text{Range}(L_A)$.

(b) Prove or give a counterexample to the following statement: If the coefficient matrix of a system of $m$ linear equations in $n$ unknowns has rank $m$, then the system has a solution.

#2 Let $A$ be an $m \times n$ matrix with rank $m$. Prove that there exists an $n \times m$ matrix $B$ such that $AB = I_m$.

#3 Let $W$ be the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of the symmetric $2 \times 2$ matrices. The set

$$S = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \right\}$$

generates $W$. Find a subset of $S$ that is a basis of $W$. 

#4 Let $T$ be the linear operator on $M_{2\times 2}(\mathbb{R})$ defined by $T(A) = A^t$.

(a) Show that $\pm 1$ are the only eigenvalues of $T$.

(b) Describe the eigenvectors corresponding to each eigenvalue of $T$.

(c) Find an ordered basis $\beta$ for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

#5 For each of the following matrices $A \in M_{n\times n}(\mathbb{R})$, test $A$ for diagonalizability, and if $A$ is diagonalizable, find an invertible matrix $Q$ and a diagonal matrix $D$ such that $Q^{-1}AQ = D$:

(a) \[
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 & 3 \\
3 & 1
\end{pmatrix}
\]