(1) (a) Suppose that $v$ is the three-vector $[1 0 1]$. Find the matrix $A$ for orthogonal projection onto the span of $v$. The formula $A = vv^T/vv^T$ gives the matrix $(1/2) \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. (b) Check explicitly that $A^2 = A$. (c) Find the eigenvalues and eigenvectors of $A$. The eigenvalues are $1, 0$ with eigenvectors the span of $[1 0 1]$, $[0 1 0]$ for $\lambda = 1$ and the span of $[1 0 1], [0 1 0]$ for $\lambda = 0$ since these vectors span the span of $v$ and its orthogonal complement respectively. The dot products of the rows of $A$ with the columns of $A$ give the matrix $A$ again.

(2) Let $V$ be an inner product space. Show that $P_W : V \rightarrow V$ is orthogonal projection onto a subspace $W$, then $P_W$ satisfies the relation $\langle P_W v_1, v_2 \rangle = \langle v_1, P_W v_2 \rangle$. Substituting the formula $v_2 = P_W v_2 + P_W^\perp v_2$ for the projections gives $\langle P_W v_1, v_2 \rangle = \langle P_W v_1, P_W v_2 + P_W^\perp v_2 \rangle = \langle P_W v_1, P_W v_2 \rangle$. Now the same idea gives $\langle v_1, P_W v_2 \rangle = \langle P_W v_1 + P_W^\perp v_1, P_W v_2 \rangle = \langle P_W v_1, P_W v_2 \rangle$. So $\langle P_W v_1, v_2 \rangle = \langle v_1, P_W v_2 \rangle$.

(3) Show an isomorphism $T : V \rightarrow V$ is an orthogonal transformation iff for any orthonormal set $B$, the image $T(B)$ is also orthonormal. Suppose $T$ is an orthogonal transformation. Let $B$ be an orthonormal set. Let $Tv, Tw \in T(B)$. Then $(Tv, Tw) = \langle v, w \rangle$, since $T$ is orthogonal, which equals 0 if $v \neq w$, and equals 1 if $v = w$, since $B$ is orthonormal. So $T(B)$ is orthonormal. Conversely, suppose for any orthonormal set $B$ the image $T(B)$ is also orthonormal. Let $v \in V$ be non-zero. Then $v/\|v\|$ is a unit vector and so $B = \{v/\|v\|\}$ is an orthonormal set. So by assumption $T(B)$ is orthonormal, that is, $\|T(v)/\|v\|\| = \|Tv/\|v\|\| = 1$. So $\|Tv\| = \|v\|$. Since this also holds if $v = 0$, it holds for any vector $v \in V, T$ is orthonormal.

(4) Find the orthogonal diagonalization of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, that is, find an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $A = QDQ^T$. (Hint: find an orthogonal eigenbasis and use it to form $Q$. 

(5) True or false: $T : C^\infty([-1,1]) \rightarrow C^\infty([-1,1]), (Tf)(x) = f(-x)$ is an orthogonal transformation. Prove your answer. True, since for any smooth functions $f, g$ we have $\langle Tf, Tg \rangle = \int_{-1}^{1}(Tf)(x)(Tg)(x)dx = \int_{-1}^{1} f(-x)g(-x)dx = \int_{-1}^{1} f(x)g(x)dx = \int_{-1}^{1} f(x)g(x)dx = \langle f, g \rangle$. 

(6) Let $A$ be an $n \times n$ matrix. For each eigenvalue $\lambda$ prove that the generalized eigenspace $\tilde{E}_{\lambda} = \{v \exists k, (A - \lambda I)^k v = 0\}$ is a subspace of $\mathbb{R}^n$. We have to check the definition of subspace, i.e., that the set contains 0 and is closed under linear combinations. First note that $(A - \lambda I)0 = 0$ so 0 is $\tilde{E}_{\lambda}$. Next suppose that $v, w \in \tilde{E}_{\lambda}$ and $c, d$ are scalars. Then for some $k, l (A - \lambda I)^k v = 0$ and $(A - \lambda I)^l w = 0$. Then $(A - \lambda I)^{k+l}(cv + dw) = c(A - \lambda I)^k (A - \lambda I)^l w = 0$. So $cv + dw \in \tilde{E}_{\lambda}$ so $\tilde{E}_{\lambda}$ is closed under linear combinations.

(7) Find a basis of generalized eigenvectors for the following matrices. In each case write down the Jordan form. Omitted.

(8) Find all possible $2 \times 2$ Jordan matrices $A$ satisfying $A^3 = A$ and $A^2 \neq A$. Justify your answer. Since any eigenvalue $\lambda$ satisfies $\lambda^3 v = \lambda v$ with $v$ non-zero, we must have $\lambda^3 = \lambda$ so $\lambda$ has possible eigenvalues $-1, 1, 0$. Since $A = SJS^{-1}$ and $A^3 = A$, we have $SJ^3S^{-1} = SJS^{-1}$ so $J^3 = J$. Similarly, $J^2 \neq J$. If $J$ has as size two or bigger Jordan block then $J^3 \neq J$, so $J$ is diagonalizable with at least one of the eigenvalues equal to $-1$ (since otherwise we would have $J^2 = J$). So the possibilities are $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. 

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