Problem Set 9, Math 350, Fall 2017

(1) (a) Suppose that $v$ is the three-vector $[1\ 0\ 1]$. Find the matrix $A$ for orthogonal projection onto the span of $v$. (b) Check explicitly that $A^2 = A$. (c) Find the eigenvalues and eigenvectors of $A$.

(2) Let $V$ be an inner product space. Show that if $P_W : V \rightarrow V$ is orthogonal projection onto a subspace $W$, then $P_W$ satisfies the relation $\langle P_W v_1, v_2 \rangle = \langle v_1, P_W v_2 \rangle$. (Hint: write out $v_2$ as a sum of its projections and similarly for $v_1$.)

(3) Show an isomorphism $T : V \rightarrow V$ is an orthogonal transformation iff for any orthonormal set $B$, the image $T(B)$ is also orthonormal.

(4) Find the orthogonal diagonalization of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, that is, find an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $A = QDQ^T$. (Hint: find an orthogonal eigenbasis and use it to form $Q$.)

(5) True or false: $T : C^\infty([-1, 1]) \rightarrow C^\infty([-1, 1]), (Tf)(x) = f(-x)$ is an orthogonal transformation. Prove your answer.

(6) Let $A$ be an $n \times n$ matrix. For each eigenvalue $\lambda$ prove that the generalized eigenspace $\tilde{E}_\lambda = \{v | \exists k, (A - \lambda I)^k v = 0\}$ is a subspace of $\mathbb{R}^n$.

(7) Find a basis of generalized eigenvectors for the following matrices. In each case write down the Jordan form.

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, (b) $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$,

(c) $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(8) Find all possible $2 \times 2$ Jordan matrices $A$ satisfying $A^3 = A$ and $A^2 \neq A$. Justify your answer.