Problem Set 8, Math 350, Fall 2017

(1) (a) Two square matrices A, B of the same size are said to *commute* if $AB = BA$. Show that if A, B commute then $A + B, B$ commute as well.

(b) Let A, B be commuting matrices. Show that if v is an eigenvector of A, then Bv is also an eigenvector with the same eigenvalue.

(c) Show that if A, B commute then any eigenvalue for $A + B$ is the sum of an eigenvalue for A and an eigenvalue for B. (Hint: If $E_\lambda(A + B)$ is an eigenspace for $A + B$, then by parts (a) and (b) we have that $BE_\lambda(A + B) \subseteq E_\lambda(A + B)$. Let v be an eigenvector for B in $E_\lambda(A + B)$ ....)

(d) The *discrete Laplacian* (or discrete second derivative) is the operator $L : R^n \to R^n$ defined by $(Lv)_k = (v_{k+1} - v_k) - (v_k - v_{k-1})$ where the indices are taken mod n (so $v_{n+1} := v_1$). Find the matrix of T with respect to the standard basis. (Your answer may use the $\ldots$ notation.)

(e) Find the eigenvalues of the discrete Laplacian from part (d). (Hint: Write the matrix for T as a sum of commuting matrices whose eigenvalues are easy to find and use part (c).)

(2) (a) A square matrix A is *stochastic* if its entries are non-negative and the columns sum to 1. Show that the product of two stochastic matrices of the same size is stochastic.

(b) Show that for any stochastic $n \times n$ matrix A and standard basis vector e_j, the entries of $A^k e_j$ are between 0 and 1. Use this to show that for any n vector $v = [v_1\ v_2\ \ldots\ v_n]$ each entry of $A^k v$ is between $-(|v_1| + \ldots + |v_n|)$ and $+(|v_1| + \ldots + |v_n|)$.

(c) Using (b) show that the eigenvalues λ of a stochastic matrix satisfy $|\lambda| \leq 1$.