Problem Set 2 Answers, Math 350, Woodward, Fall 2017

(1) Prove that a nonempty subset \( W \) of a vector space \( V \) is a subspace iff \( \text{span}(W) = W \). Ans: Since the statement is an if and only if, we have to prove both directions.

\( \Rightarrow \) First suppose \( W \) is a subspace of \( V \). We claim \( \text{span}(W) = W \), that is, \( \text{span}(W) \subseteq W \) and \( \text{span}(W) \supseteq W \).

\( \subseteq \) To prove \( \text{span}(W) \subseteq W \), we must show that for all \( x \) if \( x \in \text{span}(W) \) then \( x \in W \). Suppose \( x \in \text{span}(W) \). By definition \( x \) is a linear combination of elements of \( W \) so \( x = c_1w_1 + \ldots + c_kw_k \) for some scalars \( c_1, \ldots, c_k \) and vectors \( w_1, \ldots, w_k \in W \). Since \( W \) is closed under scalar multiplication \( c_iw_i \in W \) for each \( i = 1, \ldots, k \). Since \( W \) is closed under vector addition, \( c_1w_1 + \ldots + c_kw_k = x \in W \). So \( \text{span}(W) \subseteq W \).

\( \supseteq \) Conversely, suppose \( x \in W \). Then \( x = 1x \) is a linear combination of elements of \( W \), so \( x \in \text{span}(W) \). Since this holds for all \( x \), \( \text{span}(W) \supseteq W \).

\( \Leftarrow \) For the reverse direction, suppose \( \text{span}(W) = W \). We claim \( W \) is a subspace, that is, \( 0 \in W \) and \( W \) is closed under vector addition and scalar multiplication.

(SS1) Since \( W \) is non-empty, \( W \) contains some \( x \). Since \( W \) is closed under scalar multiplication \( 0x \in W \). But \( 0x = 0 \). So \( 0 \in W \).

(SS2) Suppose \( x, y \in W \). Then \( x + y \) is a linear combination of elements of \( W \), so \( x + y \in \text{span}(W) = W \).

(SS3) Suppose \( x \in W \) and \( c \in F \). Then \( cx \) is a linear combination of elements of \( W \) so \( cx \in \text{span}(W) = W \).

(2) Prove that the polynomials \( 1, x, x^2 \) are linearly independent in the space of functions from \( \mathbb{R} \) to \( \mathbb{R} \). Ans: Suppose that there is a relation of the form \( a(1) + b(x) + c(x^2) = 0 \), as functions of \( x \). Then the equality holds for all \( x \), in particular, \( x = 0 \). So \( a = 0 \), and thus \( bx + cx^2 = 0 \). Taking \( x = 1 \) gives \( b + c = 0 \), while taking \( x = -1 \) gives \( -b + c = 0 \). Adding the equations gives \( 2c = 0 \), so \( c = 0 \) and also \( b = 0 \). Thus the functions are independent.

(3) Find a basis for the space of real polynomials \( \{ f(x) = ax^3 + bx^2 + cx + d \mid f(1) = 0 \} \). Ans: The constraint on the value of \( f \) gives the equation \( a + b + c + d = 0 \). The corresponding augmented matrix \( \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \end{bmatrix} \) is already in reduced row echelon form, so \( a \) is the bound variable and \( b, c, d \) are the free variables; the general form of the solution is therefore \( f(x) = (b - c - d)x^3 + bx^2 + cx + d \). Plugging in 1 for each free variable and 0 for the remaining free variables gives a basis for the space of solutions the functions \( \{-x^3 + x^2, -x^2 + x, -x^2 + 1\} \) as a basis.

(4) Find a basis for the vector space of real polynomials of arbitrary degree, and prove that your answer is a basis. [Warning: linear independence is a bit tricky.] Show that this space is infinite dimensional. Ans: We claim that \( \{1, x, x^2, x^3, \ldots\} \) is a basis. First, these vectors span the space: Any polynomial is of the form \( f(x) = a_0 + a_1x + \ldots + a_kx^k \) by definition, and so spanned by \( 1, \ldots, x^k \) for some \( k \). On the other hand, we claim these vectors are independent. Suppose that \( a_0 + a_1x + \ldots + a_kx^k = 0 \) for some scalars \( a_1, \ldots, a_k \) with some \( a_i \) non-zero. We may assume that \( a_k \neq 0 \), by decreasing \( k \) if necessary. But differentiating \( k \) times gives \( k!a_k = 0 \), so \( a_k = 0 \), a contradiction. Thus the functions are independent. From part (c) in class, the dimension of any vector space is at least the size of any independent set. Since the vectors space contains linearly independent subsets of infinite size, the dimension is infinite.

(5) Show that if \( W \) is a subspace of a finite-dimensional vector space \( V \) and \( \dim(W) = \dim(V) \) then \( V = W \). Ans: Suppose that \( W \) is a subspace of \( V \) and \( \dim(W) = \dim(V) \). Then \( W \) has a basis \( \{w_1, \ldots, w_n\} \) for some \( n \), and \( V \) has a basis \( \{v_1, \ldots, v_n\} \). We claim \( V = W \), that is, \( V \subseteq W \) and \( W \subseteq V \). The second one is by definition. For the first, it suffices to show that each \( v_i \) lies in \( W \). Suppose otherwise, so that some \( v_i \notin W \). Then \( \{w_1, \ldots, w_n, v_i\} \) is a linearly independent subset of \( V \), and \( \{v_1, \ldots, v_n\} \) spans \( V \). This contradicts what we showed in class, which was that the size of any linearly independent subset is less than or equal to the size of a spanning subset.

(6) Find a basis for the space of real skew-symmetric matrices of size \( n \). What is the dimension of the space? Ans. The general form of a real skew-symmetric matrix is \( A = (a_{ij}) \) where \( a_{ij} = -a_{ji} \), that is, \( A = \begin{bmatrix} 0 & a_{12} & \ldots & a_{1n} \\ -a_{12} & 0 & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \ldots & 0 \end{bmatrix} \). The matrix \( A \) can be written as \( \sum_{i<j} a_{ij}(E_{ij} - E_{ji}) \) where \( E_{ij} \) is the \( ij \)-th elementary matrix whose \( ij \)-th entry is 1 and all other entries are 0. So the matrices \( B_{ij} = E_{ij} - E_{ji} \) for \( i < j \) span the space. They are also linearly independent because if \( \sum a_{ij}B_{ij} = 0 \) then in particular the \( ij \)-th entry shows \( a_{ij} = 0 \). The number of such matrices is the number of upper-triangular entries in an \( n \times n \) matrix which is \( (n-1) + (n-2) + \ldots + 1 = n(n-1)/2 \).