(1) Find the space of polynomials \( f(x) = ax^2 + bx + c \) satisfying \( f(-1) = -1, f'(1) = -1 \). Ans: The constraints give the linear system \( a - b + c = -1, -2a + b = -1 \). The system can be solved by Gaussian elimination: Adding twice the first equation to the second gives \(-b + 2c = -3\) so \( b = 2c + 3 \) and \( a = (b+1)/2 = c+2 \). So the space of solutions is \( \{ f(x) = (2 + c)x^2 + (2c + 3)x + c, c \in \mathbb{R} \} \).

(2) Show that the set \( V \) of functions \( f : S \to \mathbb{R} \) from a set \( S \) to the real numbers \( \mathbb{R} \) with addition given by \( (f + g)(x) = f(x) + g(x) \) and scalar multiplication given by \( (cf)(x) = c(f(x)) \) satisfies axiom (VS4) in the axioms for a vector space. Ans: We claim that for any \( f \in V \), there exists a \( g \in V \) such that \( f + g = 0 \), or in other words, \( f(x) + g(x) = 0 \) for all \( x \in S \). Let \( g \) be the function given by \( g(x) = -f(x), x \in S \). Then \( (f + g)(x) = f(x) + g(x) = f(x) - f(x) = 0 \) for all \( x \in S \) as desired.

(3) Show that the set of polynomials in a single real variable with addition given by \( (f + g)(x) = f(x) + g(x) \) and scalar multiplication given by \( (cf)(x) = f(cx) \) is not a vector space. Ans. To show that \( (V, +, \cdot) \) as above is not a vector space, it suffices to show that one axiom fails. We claim that the axiom VS8 fails, that is, there \( \exists f \in V, a, b \in \mathbb{R} \) such that \( (a + b)f \neq af + bf \). Indeed, take \( f(x) = x^2, a = 2 \) and \( b = 3 \). Then \( ((a + b)f)(x) := f((a + b)x) = f(5x) = 25x^2 \) but \( (af + bf)(x) = (2f)(x) + (3f)(x) = f(2x) + f(3x) = (2x)^2 + (3x)^2 = 13x^2 \). If \( x = 1 \), then \( 25x^2 \) is not equal to \( 13x^2 \), which shows that the two functions are not equal as claimed.

(4) Show that if \( V \) is a vector space and \( v \in V \) and \( c \in F \) are such that \( cv = 0 \) then either \( c = 0 \) or \( v = 0 \). [Hint: argue by cases or by contradiction.] Ans. Suppose that \( cv = 0 \). We claim that \( c = 0 \) or \( v = 0 \). If \( c = 0 \), then we are done. Otherwise if \( c \neq 0 \) then multiplying both sides by \( 1/c \) and using axiom VS6 and VS5 gives \( (1/c)(cv) = ((1/c)c)v = 1v = v \). On the other hand, we showed in class that \( (1/c)0 = 0 \). Hence \( v = 0 \) which proves the claim in this case as well.

(5) Prove that \( \{(1,1,0),(1,0,1),(0,1,1)\} \) is linearly independent over \( \mathbb{R} \) but linearly dependent over \( \mathbb{Z}_2 \). Ans. Working with the field \( \mathbb{R} \), the determinant of the resulting \( 3 \times 3 \) matrix is 2, which shows linear independence in that case. On the other hand, if the field is \( \mathbb{Z}_2 \) then \( (1,1,0) + (1,0,1) + (0,1,1) = (2,2,2) = (0,0,0) \) mod 2 which is a dependence relation, so the vectors are dependent in this case.