Math 350 Section 2 Fall 2017 Final Review Problems

(1) Find all polynomials \( f(x) \) of degree at most 3 satisfying \( f'(1) = f''(1) \).

(2) True or false: the functions 1, \( x - 1 \), \( (x - 1)^2 \) are linearly independent as functions of a real variable \( x \). Prove your answer.

(3) True or false: the set of bounded sequences \( \mathbb{N} \rightarrow [-1, 1] \) between \(-1 \) and \( 1 \) is vector space.

(4) Let \( V = W = \text{span} B \) where \( B = \{ e^x, e^{2x} \} \). Let \( T : V \rightarrow W, (Tf)(x) = f(x + 1) \). Find the matrix \( A \) of \( T \) with respect to \( B \).

(5) Let \( T \) be the linear transformation on \( P_2 \) given by \( (Tf)(x) = (f(x) - f(0))/x \). Find the matrix for \( T \) with respect to the standard basis and the range and nullspace of \( T \).

(6) Show that if \( W_1 \cap W_2 = \{ 0 \} \) are subspace of a vector space \( V \) then \( \dim(W_1 + W_2) = \dim(W_2) + \dim(W_1) \).

(7) Show that if \( T : V \rightarrow W \) is an isomorphism and \( \{ v_1, \ldots, v_k \} \) spans \( V \) then \( \{ T(v_1), \ldots, T(v_k) \} \) spans \( W \).

(8) Let \( T : V \rightarrow V \) be a linear transformation with \( T^k = I \) for some \( k \). Show that \( R(T) = V \).

(9) True/false: The row-space of a matrix \( A \) is equal to the range of the transformation \( x \mapsto Ax \).

(10) True/false: the set of polynomials \( 1, (x-1), (x-1)^2, \ldots, (x-1)^n \) is a basis for \( P_n \).

(11) (a) Find the matrix for the linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined by \( (T\mathbf{v})_k = v_{k+2} \) (where the indices are taken mod \( n \)) for \( n = 7 \). (b) Find the inverse of \( T \). (c) Find the eigenvalues of \( T \).

(12) True/false: The reduced row echelon form of a matrix has the same nullspace as the original matrix. Prove your answer.

(13) True/false: The eigenvalues of an orthogonal matrix \( \lambda \) satisfy \( |\lambda| = 1 \). Prove your answer.

(14) Prove that if \( B \) is related to \( A \) by switching the first two rows, then \( \det(B) = -\det(A) \).

(15) True/false: The map \( (f, g) \mapsto -\int_0^1 f(x)g'(x)dx \) defines an inner product on the space of continuous functions from 0 to 1.

(16) Prove that if \( v_1, v_2 \) are orthonormal and \( v \) lies in the span of \( v_1, v_2 \) then \( v = \langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2 \).

(17) Find the linear function \( a + bx \) that best approximates the function \( f(x) = \sin(x) \) on the interval \([0, \pi/2]\) with respect to the standard inner product on \( C^\infty([0, \pi/2]) \). (That is, find the function \( a + bx \) closest to \( \sin(x) \).)

(18) Prove that if \( v, w \) are orthogonal then \( v \) is the closest vector to \( v + w \) in \( \text{span}(v) \).

(19) (a) Suppose that the reduced row echelon form of a matrix is \[
\begin{bmatrix}
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] If the second and third columns of the matrix are \[
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}
\] and \[
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}
\] respectively, what is the matrix? (b) Find a basis for the null-space of the matrix? (c) Find a basis for the column-space of the matrix.

(20) Find the matrix for projection onto the span of the vector \( [1 \ 2 \ 3] \).

(21) Prove that any projection matrix is diagonalizable.

(22) Show if a projection matrix is orthogonal, then it is the identity transformation.

(23) Find the orthogonal diagonalization of the matrix \( A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \), that is, find an orthogonal matrix \( Q \) and a diagonal matrix \( D \) such that \( A = QDQ^T \). (Hint: find an orthogonal eigenbasis and use it to form \( Q \).)

(24) True/false: \( T : C^\infty([-1, 1]) \rightarrow C^\infty([-1, 1]), (Tf)(x) = f'(x) \) is an orthogonal transformation. Prove your answer.

(25) Let \( A \) be an \( n \times n \) matrix. For each eigenvalue \( \lambda \) prove that \( A\mathcal{E}_\lambda \subseteq \mathcal{E}_\lambda \), where \( \mathcal{E}_\lambda \) is the generalized eigenspace.

(26) Find a basis of generalized eigenvectors for the following matrix and the Jordan .form.
\[
A = \begin{bmatrix}
2 & 0 & 0 \\
1 & 2 & 0 \\
0 & 1 & 2
\end{bmatrix}
\]

(27) Find all possible \( 3 \times 3 \) Jordan matrices \( A \) satisfying \( A^3 = 0 \) but \( A^2 \neq 0 \). Justify your answer.