Lab 5: A Nonlinear System

This Maple lab is based in part on earlier versions prepared by Professors R. Falk and R. Bumby of the Rutgers Mathematics department.

Introduction. In this lab we use Maple to study the phase plane of an autonomous nonlinear system of two differential equations, i.e., a system of the form

$$x'(t) = F(x, y), \qquad y'(t) = G(x, y). \tag{1}$$

The distinguishing feature of an autonomous system is that the expressions defining the functions F and G do not contain the independent variable t. This allows many properties of the solutions to be studied using the curves, called *trajectories*, that show the paths in the xy plane followed by the solutions. (It is an easy exercise to show that if an initial condition is on a trajectory, then the whole solution follows that trajectory). The Maple command `DEplot` may be used to draw trajectories and direction fields for such systems.

Please obtain the seed file from the web page and save it in your directory on eden, then prepare the Maple lab according to the instructions and hints in the introduction to Lab 0. Turn in only the printout of your Maple worksheet, after removing any extraneous material and any errors you have made.

0. Setup. As usual, the seed file begins with commands which load the required Maple packages: `with(plots):`, `with(DEtools):`, and `with(LinearAlgebra):`.

1. An autonomous nonlinear system: a. The equation and its critical points.

We will study the nonlinear system

$$\begin{aligned} x' &= -x^2 + xy + y + 1 = (x+1)(y-x+1) \\ y' &= -x^2 - xy + 5x + 4y - 4 = (x+y-1)(-x+4) \end{aligned} \tag{2}$$

The equilibrium solutions are $[x = -1, y = 2]$, $[x = 1, y = 0]$, and $[x = 4, y = 3]$. Maple can obtain these by using the instructions

```
F:=-(x+1)*(x-y-1);
G:=(x+y-1)*(-x+4);
critpts:=solve(F,G,x,y);
```

which are included in the seed file. Also included are instructions to define the differential equations:

```
dex:=diff(x(t),t)=eval(F,{x=x(t),y=y(t)});
dey:=diff(y(t),t)=eval(G,{x=x(t),y=y(t)});
```

For later convenience we have defined F and G to depend on the variables x and y, but in the differential equations we must write these variables as $x(t)$ and $y(t)$; the `eval` command makes this substitution.

b. The direction field and nullclines. The next instructions establish an appropriate range for the independent variable and a useful viewing window, and construct a plot of the direction field of this system. The plot consists of small arrows pointing the way of the trajectories in the square $-5 \le x \le 5$, $-5 \le y \le 5$.

```
trange := -5..5:  window:=x=-5..5,y=-5..5:
df:=DEplot([dex,dey],[x(t),y(t)], trange, window,color=GREEN):
```

(Note the colon at the end of the instruction defining the plot, which suppresses output of the plot structure). The `color` option is used to give a better view of the *nullclines* and solution curves to be added later. If you like, you can test different values of the option `dirfield`, as in Lab 2, and customize the value for the plots in this project. A value should be chosen that allows individual arrows to be identified while including a sufficient number of arrows. To show this plot, you need only enter its name `df1;` after defining it. The result *should not be left in the worksheet*, since we will include this as part of a `display` command in the next subsection.

Nullclines. The points where the direction field is *horizontal* (characterized by $dy/dt = 0$) or *vertical* (characterized by $dx/dt = 0$) form curves called *nullclines*. In many cases, these curves provide useful information about the behavior of trajectories without the excessive detail of a direction field.

For the system (2), the factored form of the expressions for dx/dt and dy/dt allows a simple description of the nullclines. In particular, since $x' = (x+1)(y-x+1)$, the slope field is vertical along the curves $x = -1$ and $y = x - 1$. Since $x = -1$ is a vertical line, it is best to plot these curves parametrically; such a plot can be constructed using

```
dh:=plot([[-1,t,t=trange],[t,t-1,t=trange]],window,color=CORAL):
```

Construct a similar instruction to produce a plot `dh` showing where the slope field is horizontal; to be definite, use `color=BLUE`. Now obtain a `display` (including a title) combining the slope field and both sets of nullclines, using the following instruction in the seed file:

```
display(df,dh,dv,title="Slope field and nullclines:  Nonlinear equation");
```

The equilibrium points should be seen as points lying on one nullcline of each color. Note that the nullclines usually *cut across the arrows* in the slope field since they are *not* trajectories of the system; the nullcline $x = -1$ is exceptional.

Discussion: For each of the three equilibrium points found in part **a**, give explicitly the equations of the vertical and the horizontal nullclines which cross at that point.

c. Trajectories. To study the behavior of the system and the nature and stability of equilibrium points, it is also useful to construct a plot of the phase plane which shows several trajectories as well as the direction field. In Lab 4, we did this by finding the *exact* solution to the equation. Since this is in general not possible for nonlinear equations, we use here the *numerical methods* that are part of the `DEplot` command. To produce each trajectory one specifies an initial condition—i.e., a point in the phase plane that is to correspond to $t = 0$—and asks Maple to produce the trajectory through that point; to get a significant part of the full trajectory one solves over a range of the independent variable, t, which includes both positive and negative values.

To obtain good choices of initial conditions, leading to a set of trajectories which illustrate the important features of the phase plane, requires some trial and error. The seed file contains a list of six initial conditions, together with a command to produce a plot of the phase plane containing the direction field and the corresponding six trajectories:

```
inits:=[[x(0)=3,y(0)=4],[x(0)=2,y(0)=-1],[x(0)=5,y(0)=0],
[x(0)=1,y(0)=0.2],[x(0)=-.5,y(0)=.5],[x(0)=-2,y(0)=3]]:
DEplot([dex,dey],[x(t),y(t)],t=trange,inits,window,color=GREEN,
linecolor=[RED,BLUE,BROWN,PLUM,CORAL,BLACK],thickness=2,stepsize=0.005,
title=``Trajectories:  Nonlinear equation'');
```

The choice of stepsize here seems to work well, but you can experiment with other possible choices. Include the resulting plot in your final worksheet.

Discussion: By examining the phase plane plot, you should be able to decide the *type* of each of the three critical points: stable or unstable node, saddle point, or stable or unstable spiral. Give your conclusions for each point in the *discussion* section.

Discussion: In the second *discussion* section you should describe any interesting properties of each of the six trajectories shown. In particular, *for each one*, discuss whether or not that trajectory appears to approach a critical point—and if so, which one—as $t \to \infty$ and as $t \to -\infty$. In some cases you will not be able to determine this from your graph, and it is perfectly acceptable to say so. In your discussion you should identify the various trajectories by color.

d. Linearization. The type and stability of the critical points can be determined by examining the eigenvalues of the corresponding linear system. The matrix of the linearization of the system (1) at the critical point (x_c, y_c) is

$$A := \begin{pmatrix} F_x(x_c, y_c) & F_y(x_c, y_c) \\ G_x(x_c, y_c) & G_y(x_c, y_c) \end{pmatrix}.$$

Maple can find this matrix in a two step process, first constructing the matrix of partial derivatives (which need be done only once), then evaluating it at the critical point (done separately for each point). Instructions for finding A for general x and y and then finding A_1, its value at the critical point $(-1, 2)$, and the eigenvalues of A_1, are in the seed file:

```
A := Matrix([[diff(F,x),diff(F,y)],[diff(G,x),diff(G,y)]]);
A1 := eval(A,{x=-1,y=2});
Eigenvalues(A1);
```

Use Maple to find the matrices A_2 and A_3, and their eigenvalues, corresponding to the critical points on the x-axis and in the first quadrant, respectively.

Discussion: In a *discussion section*, for each critical point, describe what the eigenvalues you found imply about the type of that point, and compare the answer with the one you found in part **c**. (The answers should agree!)

The phase plane near the critical points. In sections **e**, **f**, and **g** we sketch the phase plane for the true system, and separately for its linearization, near each of the three critical points. We use a square window, 0.4 units by 0.4 units, centered at the critical point, and as initial conditions the four points $(x_c \pm 0.1, y_c)$ and $(x_c, y_c \pm 0.1)$.

Discussion: In each section you should produce two plots. Include also a discussion comparing the plots for the true and linearized systems: what similarities do you see? What differences?

e. The phase plane near $(-1, 2)$. For this case the seed file contains the commands needed to produce the two plots. The commands needed to graph the phase plane of the nonlinear system near $(-1, 2)$ are

```
trange1 := -3..3:  window1 := x=-1.2..-0.8,y=1.8..2.2:
inits1:=[[x(0)=-1,y(0)=2.1],[x(0)=-1,y(0)=1.9],[x(0)=-1.1,y(0)=2],
[x(0)=-0.9,y(0)=2]]:
DEplot([dex,dey],[x(t),y(t)],t=trange1,inits1, window1,color=GREEN,
linecolor=[RED,BLUE,BROWN,BLACK],thickness=2,stepsize=0.002,
title=``Phase plane near (-1,2):  nonlinear system'');
```

For the linearized system the commands are

```
F1:=4*u;
G1:=5*(u+v);
```

```
dex1:=diff(x(t),t)=eval(F1,u=x(t)+1,v=y(t)-2);
dey1:=diff(y(t),t)=eval(G1,u=x(t)+1,v=y(t)-2);
DEplot([dex1,dey1],[x(t),y(t)],t=trange1,inits1, window1,color=GREEN,
linecolor=[RED,BLUE,BROWN,BLACK,thickness=2,stepsize=0.002,
title=``Phase plane near (-1,2):  linearized system'');
```

Note that here:

(i) The functions F_1 and G_1 are obtained from the matrix A_1 found in part **d**.

(ii) In defining the linear differential equations we make the substitutions $u = x - x_c = x + 1$ and $v = y - y_c = y - 2$, so that the resulting phase portrait is centered (in the xy plane) at the point $(-1, 2)$. This provides for easy comparison with the nonlinear case.

Discussion: In your *discussion*, explain why this nonlinear system and linear system share a pair of straight line trajectories.

f. The phase plane near $(1, 0)$**.** Modify the above commands to produce plots of the phase plane for the nonlinear and linear systems near $(1, 0)$. Include a fifth trajectory, colored CORAL, with initial condition $[x(0) = 0.8, y(0) = 0.1]$. This initial point is of the form $(1, 0) + 0.1(\xi_1, \xi_2)$, where $\xi = (-2, 1)$ is one of the eigenvectors for the linear problem (found using the command `Eigenvectors(A2)`). The corresponding trajectory is a straight line in the linear system, but not in the nonlinear system.

Discussion: In your *discussion*, explain why this (coral) trajectory looks so different in the two plots, while the other trajectories are similar.

g. The phase plane near $(4, 3)$**.** Modify the above commands to produce plots of the phase plane for the nonlinear and linear systems near $(4, 3)$, and again compare the plots in your *Discussion.*

End of Lab 5