TANGENT AND SECANT $q$-CALCULUS: AND $(t, q)$-CALCULUS

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Based on the paper

“The \((t, q)\)-analogs of secant and tangent numbers”

jointly written with Guo-Niu Han
$(t, q)$-analog: the order matters!

For $q, t$-analogs and $(q, t)$-analogs see

Haiman-Woo (2007)

Reiner-Stanton (2009), respectively.
SUMMARY

One of the Gian-Carlo Rota legacies
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Standard $q$-notations
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The trick of the graded form
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$(t, q)$-analogs of secant and tangent
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The Euler-Roselle positivity problem
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One of the Gian-Carlo Rota legacies

Standard $q$-notations

The trick of the graded form

$(t, q)$-analogs of secant and tangent
ONE OF THE GIAN-CARLO ROTA LEGACIES

Every mathematician has only a few tricks in his pocket.
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perhaps very few!

A single one in this lecture.
STANDARD $q$-NOTATIONS

The $q$-ascending factorial: $$(\omega; q)_0 := 1 \text{ and for } k \geq 1$$

$$(\omega; q)_k := (1 - \omega)(1 - \omega q) \cdots (1 - \omega q^{k-1});$$
The \( q \)-ascending factorial: \( (\omega; q)_0 := 1 \) and for \( k \geq 1 \)

\[
(\omega; q)_k := (1 - \omega)(1 - \omega q) \cdots (1 - \omega q^{k-1});
\]

the \( q \)-binomial coefficient

\[
{\binom{n}{k}}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} \quad (0 \leq k \leq n);
\]
STANDARD $q$-NOTATIONS

The $q$-ascending factorial: $(\omega; q)_0 := 1$ and for $k \geq 1$

$$(\omega; q)_k := (1 - \omega)(1 - \omega q) \cdots (1 - \omega q^{k-1});$$

the $q$-binomial coefficient

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} \quad (0 \leq k \leq n);$$

the $q$-analogs of the integers and factorials

$$[n]_q := \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1};$$

$$[n]!_q := \frac{(q; q)_n}{(1 - q)^n} = [n]_q [n - 1]_q \cdots [1]_q.$$
THE TRICK OF THE GRADED FORM

Consider:

\[ G(1, u) = \sum_{n} A_n(1, 1) u^n / n! \]

\( G(1, u) \) the exp. g.f. for the sequence \( (A_n(1, 1)) \);
Consider:

\[
G(1, u) = \sum_n A_n(1, 1) \frac{u^n}{n!}
\]

\[
G(q; u) = \sum_n A_n(1, q) \frac{u^n}{(q; q)_n}
\]

\(G(1, u)\) the exp. g.f. for the sequence \((A_n(1, 1))\);

\(G(q; u)\) the \(q\)-factorial g.f. for the sequence \((A_n(1, q))\);
THE TRICK OF THE GRADED FORM

The horizontal arrow makes sense

\[
G(1, u) = \sum_n A_n(1, 1) \frac{u^n}{n!} \quad \rightarrow \quad G(q; u) = \sum_n A_n(1, q) \frac{u^n}{(q; q)_n}
\]

if

(1) \[ \lim_{q \to 1} G(q; u(1 - q)) = G(u). \]
THE TRICK OF THE GRADED FORM

Now, let \((G_r(q; u))_{(r \geq 0)}\) be a sequence of \(q\)-series such that

\[
\lim_{r \to \infty} G_r(q; u) = G(q; u).
\]

Form the graded form \(\sum_{r \geq 0} t^r G_r(q; u)\).
THE TRICK OF THE GRADED FORM

Now, let \((G_r(q; u))_{(r \geq 0)}\) be a sequence of \(q\)-series such that

\[
\lim_{r} G_r(q; u) = G(q; u).
\]

Form the graded form \(\sum_{r \geq 0} t^r G_r(q; u)\).

Then

\[
(1 - t) \sum_{r \geq 0} t^r G_r(q; u) \bigg|_{t = 1} = G(q; u).
\]
THE TRICK OF THE GRADED FORM

The relation

\[(3) \quad (1 - t) \sum_{r \geq 0} t^r G_r(q; u) \bigg|_{t = 1} = G(q; u).\]

gives a sense to the vertical arrow

\[
\sum_r t^r G_r(q; u) = \sum_n A_n(t, q) \frac{u^n}{(t; q)_{n+1}}
\]

\[
G(1, u) = \sum_n A_n(1, 1) \frac{u^n}{n!} \quad \rightarrow \quad G(q; u) = \sum_n A_n(1, q) \frac{u^n}{(q; q)_n}
\]
(t, q)-ANALOGS OF SECANT AND TANGENT

Apply this trick of the graded forms to the sequences of
the secant numbers \((E_{2n}) (n \geq 0)\)
and the tangent numbers \((T_{2n+1}) (n \geq 0)\).
\((t, q)\)-ANALOGS OF SECANT AND TANGENT

The **secant numbers** \(E_{2n}\ (n \geq 0)\) defined by

\[
\sec u = \frac{1}{\cos u} = 1 + \sum_{n \geq 1} \frac{u^{2n}}{(2n)!}E_n
\]

\[
= 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \frac{u^8}{8!}1385 + \frac{u^{10}}{10!}50521 + \cdots
\]

Sloane’s Encyclopedia A122045.
(t, q)-ANALOGS OF SECANT AND TANGENT

The tangent numbers $T_{2n+1}$ ($n \geq 0$) by

$$
\tan u = \sum_{n \geq 0} \frac{u^{2n+1}}{(2n + 1)!} T_{2n+1}
$$

$$
= \frac{u}{1!} 1 + \frac{u^3}{3!} 2 + \frac{u^5}{5!} 16 + \frac{u^7}{7!} 272 + \frac{u^9}{9!} 7936 + \frac{u^{11}}{11!} 353792 + \cdots
$$

Sloane’s Encyclopedia A000182
(\(t, q\))-ANALOGS OF SECANT AND TANGENT

Huge formulary, see:

Niels Nielsen, Traité élémentaire des nombres de Bernoulli, Paris, Gauthier-Villars 1923.
(t, q)-ANALOGS OF SECANT AND TANGENT

Work out a (t, q)-analog with

\[ G(u) = \sec u \quad \text{or} \quad \tan u. \]
$(t, q)$-ANALOGS OF SECANT AND TANGENT

Jackson (1904) introduced both

$$\sin_q(u) := \sum_{n \geq 0} (-1)^n \frac{u^{2n+1}}{(q; q)_{2n+1}};$$

$$\cos_q(u) := \sum_{n \geq 0} (-1)^n \frac{u^{2n}}{(q; q)_{2n}};$$
Jackson (1904) introduced both

\[ \sin_q(u) := \sum_{n \geq 0} (-1)^n \frac{u^{2n+1}}{(q; q)_{2n+1}}; \]

\[ \cos_q(u) := \sum_{n \geq 0} (-1)^n \frac{u^{2n}}{(q; q)_{2n}}; \]

so that \textbf{q-tangent} and \textbf{q-secant} are defined by:

\[ \tan_q(u) := \frac{\sin_q(u)}{\cos_q(u)} = \sum_{n \geq 0} \frac{u^{2n+1}}{(q; q)_{2n+1}} T_{2n+1}(q); \]

\[ \sec_q(u) := \frac{1}{\cos_q(u)} = \sum_{n \geq 0} \frac{u^{2n}}{(q; q)_{2n}} E_{2n}(q). \]
(t, q)-ANALOGS OF SECANT AND TANGENT

Have $T_{2n+1}(q)$ and $E_{2n}(q)$ been studied?

Not so much,

only in scattered papers (Stanley (1976), Andrews-Gessel (1978), ..., and a few others)

or Oberwolfach talks (Schützenberger (1975)).
(t, q)-ANALOGS OF SECANT AND TANGENT

First values:

\[ E_0(q) = E_2(q) = 1, \quad E_4(q) = q(1 + q)^2 + q^4, \]
\[ E_6(q) = q^2(1 + q)^2(1 + q^2 + q^4)(1 + q + q^2 + 2q^3) + q^{12}, \]
\[ T_1(q) = 1, \quad T_3(q) = q(1 + q), \]
\[ T_5(q) = q^2(1 + q)^2(1 + q^2)^2, \]
\[ T_7(q) = q^3(1 + q)^2(1 + q^2)(1 + q^3)(1 + q + 3q^2 + 2q^3 + 3q^4 + 2q^5 + 3q^6 + q^7 + q^8). \]
\((t, q)\)-ANALOGS OF SECANT AND TANGENT

Appropriate start:

\[
\begin{align*}
\sec u & \quad \longrightarrow \quad \sec_q(u) \\
= \sum_n E_{2n} \frac{u^{2n}}{(2n)!} & \quad = \sum_n E_{2n}(q) \frac{u^{2n}}{(q; q)_{2n}} \\
\tan u & \quad \longrightarrow \quad \tan_q(u) \\
= \sum_n T_{2n+1} \frac{u^{2n+1}}{(2n + 1)!} & \quad = \sum_n T_{2n+1}(q) \frac{u^{n}}{(q; q)_{2n+1}}
\end{align*}
\]

as

\[
\lim_{q \to 1} \sec_q(u(1 - q)) = \sec u;
\]

\[
\lim_{q \to 1} \tan_q(u(1 - q)) = \tan u.
\]
\[(t, q)\text{-ANALOGS OF SECANT AND TANGENT}\]

Find out two sequences \( (\sec_q^{(r)}(u)) \ (\tan_q^{(r)}(u)) \ (r \geq 0) \)

such that

\[
\lim_{r} \ sec_q^{(r)}(u) = \sec_q(u)
\]

and

\[
\lim_{r} \ tan_q^{(r)}(u) = \tan_q(u).
\]
(t, q)-ANALOGS OF SECANT AND TANGENT

Take

\[ \sin_q^{(r)}(u) := \sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n+1}}{(q; q)_{2n+1}} u^{2n+1}; \]

\[ \cos_q^{(r)}(u) := \sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n}}{(q; q)_{2n}} u^{2n}; \]
\((t, q)\)-ANALOGS OF SECANT AND TANGENT

Take

\[
\sin_q^{(r)}(u) := \sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n+1}}{(q; q)_{2n+1}} u^{2n+1};
\]

\[
\cos_q^{(r)}(u) := \sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n}}{(q; q)_{2n}} u^{2n};
\]

\[
\sec_q^{(r)}(u) := \frac{1}{\cos_q^{(r)}(u)};
\]

\[
\tan_q^{(r)}(u) := \frac{\sin_q^{(r)}(u)}{\cos_q^{(r)}(u)};
\]
(\(t, q\))-ANALOGS OF SECANT AND TANGENT

and verify

\[
\lim_r \sec_q^{(r)}(u) = \sec_q(u) \quad \text{and} \quad \lim_r \tan_q^{(r)}(u) = \tan_q(u).
\]
(t, q)-ANALOGS OF SECANT AND TANGENT

Then, define \( E_{2n}(t, q) \) by

\[
\sum t^r \sec_q^{(r)}(u) = \sum_n E_{2n}(t, q) \frac{u^{2n}}{(t; q)_{2n+1}}
\]

\[
\begin{align*}
\sec u &= \sum_n E_{2n} \frac{u^{2n}}{(2n)!} \\
\sec_q(u) &= \sum_n E_{2n}(q) \frac{u^n}{(q; q)_{2n}}
\end{align*}
\]
(t, q)-ANALOGS OF SECANT AND TANGENT

And define $T_{2n+1}(t, q)$ by

\[
\sum t^r \tan_q^{(r)}(u) = \sum T_{2n+1}(t, q) \frac{u^{2n+1}}{(t; q)_{2n+2}}
\]

\[
= \sum T_{2n+1}(t, q) \frac{u^{2n+1}}{(2n + 1)!}
\]

\[
= \sum T_{2n+1}(q) \frac{u^n}{(q; q)_{2n+1}}
\]
(t, q)-ANALOGS OF SECANT AND TANGENT

First values:

\[ E_0(t, q) = 1; \quad E_2(t, q) = t; \quad E_4(t, q) = t^2q(1 + 2q + q^2 + tq^3); \]
\[ E_6(t, q) = t^2q^2(1 + 2q + q^2 + tq(1 + 4q + 8q^2 + 10q^3 + 8q^4 + 4q^5 + q^6)) + t^2q^5(2 + 5q + 6q^2 + 5q^3 + 2q^4) + t^3q^{10}; \]
$(t, q)$-ANALOGS OF SECANT AND TANGENT

First values:

$T_1(t, q) = t$; $T_3(t, q) = t^2 q(1 + q)$;

$T_5(t, q) = t^2 q^2 (1 + q)(1 + tq(1 + 2q + 2q^2 + q^3) + t^2 q^6)$;

$T_7(t, q) = t^2 q^3 (1 + q)(1 + tq(2 + 5q + 7q^2 + 7q^3 + 5q^4 + 2q^5) + t^2 q^3 (1 + 4q + 10q^2 + 15q^3 + 18q^4 + 15q^5 + 10q^6 + 4q^7 + q^8) + t^3 q^8 (2 + 5q + 7q^2 + 7q^3 + 5q^4 + 2q^5) + t^4 q^{14})$. 
(\(t, q\))-ANALOGS OF SECANT AND TANGENT

Recall: \(E_{2n}(t, q)\) defined by

\[
\sum_{r \geq 0} t^r \frac{1}{\sum_{n \geq 0} (-1)^n \binom{q^r; q}{2n} u^{2n}} = \sum_{n \geq 0} E_{2n}(t, q) \frac{u^{2n}}{(t; q)_{2n+1}}.
\]

Prove that each \(E_{2n}(t, q)\) is a polynomial with positive integral coefficients.
(t, q)-ANALOGS OF SECANT AND TANGENT

Also \( T_{2n+1}(t, q) \) defined by

\[
\sum_{r \geq 0} t^r \sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n+1}}{(q; q)_{2n+1}} u^{2n+1} = \sum_{n \geq 0} T_{2n+1}(t, q) \frac{u^{2n+1}}{(t; q)_{2n+2}}.
\]

Prove that each \( T_{2n+1}(t, q) \) is a polynomial with positive integral coefficients.
(t, q)-ANALOGS OF SECANT AND TANGENT

For

\[ \sec_q(u) = \sum_{n \geq 0} E_{2n}(q) \frac{u^{2n}}{(q; q)_{2n}} \]

and

\[ \tan_q(u) = \sum_{n \geq 0} T_{2n+1}(q) \frac{u^{2n+1}}{(q; q)_{2n+1}} \]

easy: \( E_{2n}(q) \) and \( T_{2n+1}(q) \) are polynomials with positive integral coefficients:
(t, q)-ANALOGS OF SECANT AND TANGENT

For

\[ \sec_q(u) = \sum_{n \geq 0} E_{2n}(q) \frac{u^{2n}}{(q; q)_{2n}} \]

and

\[ \tan_q(u) = \sum_{n \geq 0} T_{2n+1}(q) \frac{u^{2n+1}}{(q; q)_{2n+1}} \]

easy: \( E_{2n}(q) \) and \( T_{2n+1}(q) \) are polynomials with positive integral coefficients:

Just \( q \)-mimick the differential properties of secant and tangent,

using the \( q \)-binomial theorem.
(t, q)-ANALOGS OF SECANT AND TANGENT

For $E_{2n}(t, q)$ and $T_{2n+1}(t, q)$ only use the very definitions:

$$
\sum_{r \geq 0} t^r \frac{1}{\sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n}}{(q; q)_{2n}} u^{2n}} = \sum_{n \geq 0} E_{2n}(t, q) \frac{u^{2n}}{(t; q)_{2n+1}};
$$

$$
\sum_{r \geq 0} t^r \frac{\sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n+1}}{(q; q)_{2n+1}} u^{2n+1}}{\sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n}}{(q; q)_{2n}} u^{2n}} = \sum_{n \geq 0} T_{2n+1}(t, q) \frac{u^{2n+1}}{(t; q)_{2n+2}}.
$$
(t, q)-ANALOGS OF SECANT AND TANGENT

For $E_{2n}(t, q)$ and $T_{2n+1}(t, q)$ only use the very definitions:

$$
\sum_{r \geq 0} t^r \frac{1}{\sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n}}{(q; q)_{2n}} u^{2n}} = \sum_{n \geq 0} E_{2n}(t, q) \frac{u^{2n}}{(t; q)_{2n+1}};
$$

$$
\sum_{r \geq 0} t^r \frac{\sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n+1}}{(q; q)_{2n+1}} u^{2n+1}}{\sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n}}{(q; q)_{2n}} u^{2n}} = \sum_{n \geq 0} T_{2n+1}(t, q) \frac{u^{2n+1}}{(t; q)_{2n+2}}.
$$

Recourse to combinatorial methods.

Refer to the objects counted by secant and tangent.
Following Désiré André (1881) each permutation

\[ \sigma = \sigma(1) \cdots \sigma(n) \]

is said to be **alternating** if \( \sigma(1) < \sigma(2), \ \sigma(2) > \sigma(3), \ \sigma(3) < \sigma(4) \), etc. in an alternating way.

Let \( \mathcal{T}_n \) designate the set of alternating permutations of order \( n \). Désiré André showed that

\[ \#\mathcal{T}_{2n+1} = T_{2n+1}, \quad \#\mathcal{T}_{2n} = E_{2n}. \]
(t, q)-ANALOGS OF SECANT AND TANGENT

With “inv” being the number of inversions

$q$-mimick Désiré André’s derivation:

$$\sum_{\sigma \in T_n} q^{inv} \sigma = E_n(q) = E_n(1, q)$$

if \( n \) even and \( = T_n \) if \( n \) odd.
(t, q)-ANALOGS OF SECANT AND TANGENT

With “inv” being the number of inversions

q-mimick Désiré André’s derivation:

\[ \sum_{\sigma \in T_n} q^{\text{inv} \sigma} = E_n(q) = E_n(1, q) \]

if \( n \) even and \( = T_n \) if \( n \) odd.

But what to do with the variable “t”?

Look for other statistics.
(t, q)-ANALOGS OF SECANT AND TANGENT

For each permutation $\sigma = \sigma(1) \cdots \sigma(n)$ let

$$\text{IDES } \sigma := \{ \sigma(i) : 1 + \sigma(i) = \sigma(j) \text{ for } 1 \leq j \leq i - 1 \};$$
For each permutation $\sigma = \sigma(1) \cdots \sigma(n)$ let

$$\text{IDES } \sigma := \{ \sigma(i) : 1 + \sigma(i) = \sigma(j) \text{ for } 1 \leq j \leq i - 1 \};$$

$$\text{ides } \sigma := \# \text{IDES } \sigma;$$
(\(t, q\))-ANALOGS OF SECANT AND TANGENT

For each permutation \(\sigma = \sigma(1) \cdots \sigma(n)\) let

\[
\begin{align*}
\text{IDES } \sigma & := \{\sigma(i) : 1 + \sigma(i) = \sigma(j) \text{ for } 1 \leq j \leq i - 1\}; \\
\text{ides } \sigma & := \# \text{IDES } \sigma; \\
\text{imaj } \sigma & := \sum_{\sigma(i) \in \text{IDES } \sigma} \sigma(i).
\end{align*}
\]
For each permutation $\sigma = \sigma(1) \cdots \sigma(n)$ let

$$\text{IDES } \sigma := \{\sigma(i) : 1 + \sigma(i) = \sigma(j) \text{ for } 1 \leq j \leq i - 1\};$$

$$\text{ides } \sigma := \# \text{IDES } \sigma;$$

$$\text{imaj } \sigma := \sum_{\sigma(i) \in \text{IDES } \sigma} \sigma(i).$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 1 & 9 & 6 & 7 & 4 & 8 & 3 \end{pmatrix}$$

$$\text{IDES } \sigma = \{1, 3, 4, 8\}; \quad \text{ides } \sigma = 4; \quad \text{imaj } \sigma = 16.$$
(t, q)-ANALOGS OF SECANT AND TANGENT

Try “imaj” as we know that

$$\sum_{\sigma \in \mathcal{T}_n} q^{\text{inv} \sigma} = \sum_{\sigma \in \mathcal{T}_n} q^{\text{imaj} \sigma}.$$
(t, q)-ANALOGS OF SECANT AND TANGENT

\begin{align*}
\text{inv} & \quad \text{imaj} \\
1 & \quad 2 \\
2 & \quad 3 \\
3 & \quad 4 \\
4 & \quad 2 \\
\end{align*}

\[ E_4(q) = q(1 + 2q + q^2 + q^3) \]
(t, q)-ANALOGS OF SECANT AND TANGENT

The most “natural” statistic that can be associated with “imaj” is “ides.”

Not quite, but “1+ides” will do the job!
(t, q)-ANALOGS OF SECANT AND TANGENT

\[ E_4(t, q) = t^2 q (1 + 2q + q^2 + tq^3) \]
(t, q)-ANALOGS OF SECANT AND TANGENT

Theorem. The coefficients of the graded forms of \(\tan_q(u)\) and \(\sec_q(u)\) are generating polynomials for the sets of alternating permutations by the pair \((1 + \text{ides}, \text{imaj})\):

\[
T_{2n+1}(t, q) = \sum_{\sigma \in T_{2n+1}} t^{1+\text{ides}\sigma} q^{\text{imaj}\sigma};
\]

\[
E_{2n}(t, q) = \sum_{\sigma \in T_{2n}} t^{1+\text{ides}\sigma} q^{\text{imaj}\sigma}.
\]

In particular, \(T_{2n+1}(t, q)\) and \(E_{2n}(t, q)\) are polynomials with positive integral coefficients.
(t, q)-ANALOGS OF SECANT AND TANGENT

Is there a computer proof?
(t, q)-ANALOGS OF SECANT AND TANGENT

Is there a computer proof?

Even a computer-aided proof?
(t, q)-ANALOGS OF SECANT AND TANGENT

Is there a computer proof?

Even a computer-aided proof?

Soon, wait for the pair: DORON - ELKHAD,

we do celebrate.