Ask Not What Doron Zeilberger Can Do For You; Ask What You Can Do For Doron

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Beatty’s Theorem, 1926; Rayleigh 1894

• For $\alpha > 0$ irrational and $1/\alpha + 1/\beta = 1$, let
  
  $A = \bigcup_{n \geq 1} \{\lfloor n\alpha \rfloor\}$, $B = \bigcup_{n \geq 1} \{\lfloor n\beta \rfloor\}$.

• Then the sets $A$, $B$ split the positive integers: $A \cap B = \emptyset$, $A \cup B = \mathbb{Z}_{\geq 1}$.

• Condition $1/\alpha + 1/\beta = 1$ is clearly necessary.

• The thm states that it’s also sufficient.

• MAA, April 2010: “The result is so astonishing and yet easily proved that we include a short proof for the reader’s pleasure.”
For any $k \in \mathbb{Z}_{\geq 1}$, the number of terms $< k$ is $\left\lfloor k/\alpha \right\rfloor + \left\lfloor k/\beta \right\rfloor$ (by irrationality of $\alpha$). This can be expressed as:

$$= k + \left\lfloor k/\alpha \right\rfloor + \left\lfloor -k/\alpha \right\rfloor = k - 1.$$ 

Similarly, $A \cup B$ contains $k$ terms $< k+1$. Hence there is exactly one term $< k+1$ but not less than $k$; it equals $k$. 

Doron for Doron: a Pleasure Proof
Application: P, N-positions in 2-player games

N-position: a position from which the Next player can force a win.
P-position: a position from which the Previous player can win.

$\mathcal{P}, \mathcal{N}$ – set of all P-positions, N-positions, respectively.
• P– Previous player can force a win.
• N– Next player can force a win. Thus:
• Position $u \in P$ iff $F(u) \subseteq N$.
• Position $u \in N$ iff $F(u) \cap P \neq \emptyset$.
• Notice that $P$ and $N$ are not symmetric.
• In the (directed) Game Graph, $P$ is the graph kernel.
• The sets $P, N$ split $\mathbb{Z}_{\geq 1}$. Conversely, splittings into $\geq 2$ sets often induce new games.
Wythoff’s game

• Define a game on two piles of tokens:
  • take any positive number of tokens from a single pile, or
  • the same (positive) number of tokens from both piles.

• Player making last move wins.

• Then \((0,0), (1,2) \in P\).
Recursive winning strategy:
\[ a_n = \text{mex} \{a_i, b_i : 0 \leq i < n\} \quad n \geq 0, \]
\[ b_n = a_n + n. \]
Algebraic strategy:

• Let $\tau = (1+\sqrt{5})/2$, which is the solution of $1/x + 1/(x+1) = 1$; $\beta = \tau^2 = \tau + 1$.

• **Theorem.** $a_n = \lfloor n\tau \rfloor$, $b_n = \lfloor n\tau^2 \rfloor$ $n \geq 0$, and the sequences $\{a_n\}$, $\{b_n\}$ are complementary for $n \geq 1$.

• Note: $\tau = [1,1,1,1,...]$ (continued fraction expansion).

• Convergents: $p_n/q_n$, where $p_{-1} = p_0 = 1$, $p_n = p_{n-1} + p_{n-2}$ $(n \geq 1)$. 
Exotic numeration system

• Every $N \in \mathbb{Z}_{\geq 1}$ has a unique representation: $N = \sum_{i \geq 0} d_ip_i$, where the digits $d_i$ satisfy $0 \leq d_i \leq t$ ($i \geq 0$), and

• $d_i = t \Rightarrow d_{i-1} = 0$ ($i \geq 1$). Then

• $R(a_n)$ = all numbers ending in an even number of 0s, $b_n$ all numbers ending in odd number of 0s; for every $n$, $R(b_n)$ is the left shift of $R(a_n)$. 
Multi-pile Wythoff: illustration $m=3$

Take any positive number from a single pile, or $a, b, c$ from the piles s.t.

(1) $a \oplus b \oplus c = 0$. ($\oplus$ is Nim-sum; this is generalization of Wythoff.)

- Write the P-positions in the form
  $$C_j = (j, A_{n_j}, B_{n_j})_{n \geq 0}, \ 1 \leq j \leq A_{n_j} \leq B_{n_j}, \ j \ fixed.$$

  **Claim:** Under the move rule (1), Wythoff strategy is **almost** preserved:
  - $A_{n_j}, B_{n_j}$ almost split $\mathbb{Z}_{\geq 1}$
  - $A_{n_j}$ is almost mex $\{A_{i_j}, B_{i_j} : i<n\}$
  - $B_{n_j} - A_{n_j} = 1 \ \forall \ large \ n.$
Explaining “almost preservation”

• For j=1, $(1,2,k) \in \mathbb{N}$ $\forall \ k \geq 2$, since $(1,2) \in \mathcal{P}$ in Wythoff. Thus 2 cannot appear in the list of P-positions of 3-pile Wythoff.

• A small set $X$ of integers is excluded.

• How does this affect, if at all, the structure of the complementary sequences?
Two conjectures (F, 1993)

(1) For every fixed $j \geq 1$, $\exists$ integer $n_j$ and finite set $X = X^j \subset \mathbb{Z}_{\geq 0}$, s.t. $\forall \ n \geq n_j$,

- $A_n^j = \text{mex} \ (X^j \cup \{A_i^j, B_i^j : 0 \leq i < n\})$,
- $B_n^j = A_n^j + n$.

(2) For every fixed $j \geq 1$, $\exists$ integer $\gamma_j$, s.t. $\forall \ n \geq n_j$,

- $A_n^j \in \{\lfloor n\tau \rfloor - \gamma_j - 1, \lfloor n\tau \rfloor - \gamma_j, \lfloor n\tau \rfloor - \gamma_j + 1\}$,
• For approaching the conjectures, investigated a splitting system perturbed by $X$:

Recall: $a_n = \text{mex} \{a_i, b_i : 0 \leq i < n \} \quad n \geq 0$, $b_n = a_n + n$, $A = \{a_n\}_{n \geq 1}$, $B = \{b_n\}_{n \geq 1}$. Then $A = \{\lfloor n\tau \rfloor\}_{n \geq 1}$, $B = \{\lfloor n\tau^2 \rfloor\}_{n \geq 1}$. 
Let $X \subseteq \mathbb{Z}_{\geq 1}$, $X$ finite, 

$a'_n = \text{mex}_1 \{X \cup \{a'_i, b'_i : n_0 \leq i < n\}\}$,

$b'_n = a'_n + n$, $n \geq n_0$,

$A' = \{a'_n\}_{n \geq n_0}$, $B' = \{b'_n\}_{n \geq n_0}$.

Let $N = \max (X) + 1$. Then $A'$, $B'$ are $N$-upper complementary for some $n_0 \geq 1$.

Relate $A'$ to $A$, $B'$ to $B$. 
Shift sequence: \( s_n := a_n - a'_n \)

- Theorem (Krieger, F 2004). \( \exists \ p \in \mathbb{Z}_{\geq 1}, \ \gamma \in \mathbb{Z} \ s.t. \ \forall \ n \geq p, \) either \( s_n = \gamma; \) or else \( \forall \ n \geq p, \ s_n \in \{\gamma - 1, \gamma, \gamma + 1\}. \) If the latter then

- \( s_n \) assumes each of the 3 values infinitely often,

- \( s_n \neq \gamma \Rightarrow s_{n-1} = s_{n+1} = \gamma. \)

- Indices of irregular shifts can be partitioned into K subsets, each of which satisfies a linear recurrence.
More on the conjectures

• We also proved Conjecture (1) \( \Rightarrow \) Conjecture (2).

• Was also proved by Xinyu Sun 2007 with additional results.

• Zeilberger, Sun (2004) proved the 2 conjectures for \( m=3 \) and \( 1 \leq j \leq 10 \).
**Example:** \( t = 2, \, n_0 = 6, \, X = \{1,5\} \).

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The first 8000 $S$ values. The pattern we observed before continues: $s_n$ assumes only three values, 7, 8, 9, where 8 is the main value, and 7 or 9 appear more and more sparsely.
The distance between the first 60 irregular (non-8) s values. While growing larger as $k$ grows, it also maintains a very regular fractal-like pattern.
\( c = 1 \)
\( n_0 = 23 \)
\( X = \{7, 9, 10, 15, 18, 20, 21, 23, 25, 26, 27, 33, 34, 35, 41, 42, 46\} \)
Standard shift value: \( \gamma = 30 \)
Number of irregular shift sequences: \( K = 77 \)
The distance between consecutive irregular values.

\[ n_{k+1} - n_k \]
$c = 3$
$n_0 = 5$
$X = \{3, 7, 8\}$
Standard shift value: $\gamma = 5$
Number of irregular shift sequences: $K = 4$
The distance between consecutive irregular $s$ values.

$n_{k+1} - n_k$
\[\alpha = 1.7382072458106652\]
\[n_0 = 29\]
\[X = \{0, 22, 24, 30, 38, 39, 41, 44, 46\}\]
\[\beta - \alpha = 0.61643\]
Standard shift value: \(\gamma = 54\)
The distance between consecutive irregular $s$ values.
Questions

• What determines the number $K$ of irregular shift sequences $s_n$?

• For some perturbation sets $X$ get $\gamma_n = n$ for all $n$, without getting the additional two values $\gamma_n = n-1$, $\gamma_n = n+1$. Characterize those cases.

• Perturbation sets for general Beatty sequences.
Continued

• Same questions for s-fold complementarity and fractional complementarity..
• Proved the two conjectures for 3 piles and $1 \leq j \leq 10$.

• Thus if you do something for Doron, you get at least a 10-fold return. Moreover, there is the prospect of an $\infty$-return.

• They wrote: The method discussed here should be extendable to prove the conjectures for Wythoff's games with more than 3 heaps. A numerical method, instead of the symbolic one presented here, may also be developed to improve the performance...
Doron and Xinyu (contnd)

• ...We hope the result presented here would be a stepping-stone for others to finally prove the conjectures, and better yet, to provide a constructive polynomial-time winning strategy for the game.
S-fold complementarity

• Let $s \in \mathbb{Z}_{\geq 1}$. Cover every positive integer exactly $s$ times.

• Theorem (O’Bryant 2002, Larsson 2010). $\alpha$ irrational and $1/\alpha + 1/\beta = s$, $\alpha < \beta$. Let

$$A = \bigcup_{n \geq s} \{ \lfloor n\alpha \rfloor \}, \quad B = \bigcup_{n \geq 1} \{ \lfloor n\beta \rfloor \}.$$ 

• Then the sets $A, B$ s-split the positive integers: $A \cup B = s \times \mathbb{Z}_{\geq 1}$. 

Proof of $s$-fold Beatty Theorem

- O’Bryant: Generating function, Power series.
- Larsson: Combinatorial.
- Pleasure proof of AMM can be extended easily to $s$-fold complementarity.
Uspensky 1927, Graham 1963

• $\alpha_1, \ldots, \alpha_m$ positive real numbers. Suppose that $\lfloor n\alpha_1 \rfloor_{n \geq 1}, \ldots, \lfloor n\alpha_m \rfloor_{n \geq 1}$ split the positive integers. Then $m \leq 2$.

• Uspensky’s proof depends on Kronecker’s Theorem on simultaneous diophantine approximation. Graham’s is purely combinatorial.
Another Question

• Does Uspensky and Graham’s result hold also for s-fold complementarity?
• We (Hegarty, Larsson, F) conjecture that the answer is positive, excepting trivial cases.
A conjecture solved for the integers, irrationals; wide open for the rationals.

• Let $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m$, $\gamma_1, \ldots, \gamma_m$ reals, $m \geq 3$. If $\bigcup_{i=1}^{m} \lfloor n \alpha_i + \gamma_i \rfloor$ ($n \geq 1$) is a DCS, then $\alpha_i = (2^m - 1)/2^{m-i}$, $i=1, \ldots, m$ (F 1973).

• Easy to see that

$\bigcup_{i=1}^{m} \lfloor n(2^m - 1)/2^{m-i} \rfloor - 2^{i-1} + 1$, $i=1, \ldots, m$, $n \geq 1$, is indeed a DCS. Example: $m=3$. 
\[ \bigcup_{i=1}^{m} \left\lfloor n \frac{(2^m-1)}{2^{m-i}} \right\rfloor - 2^{i-1} + 1, \ i=1, \ldots, m, \ n \geq 1 \]

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Split with arithmetic sequences

• Evens and odds; evens and numbers =1 mod 4, numbers=3 mod 4...

Theorem. Suppose that \( \bigcup_{i=1}^{m} (na_i+b_i) \), \( n \geq 1, m \geq 2 \), is a DCS, \( 0 < a_1 \leq \ldots \leq a_m \) integers. Then \( a_{m-1} = a_m \).

• Proof. Consider the generating function \( \sum_{i=1}^{m} z^{b_i}/(1-z^{a_i}) = z/(1-z) \).
Mirsky, Newman, Davenport, Rado

• Proof. \( \sum_{i=1}^{m} \frac{z^{b_i}}{1-z^{a_i}} = \frac{z}{1-z} \).
  Suppose \( a_{m-1} < a_m \). Let \( \xi = \text{primitive } a_m \text{th root of unity, and let } z \to \xi \). (in Erdos 1952). Erdos asked for elementary proof.

• 1\textsuperscript{st} elementary proof: Berger, Felzenbaum, F 1986. Others followed.
Irrational case

- $\alpha > 0$ irrational, $1/\alpha + 1/\beta = 1$. Then $\lfloor n\alpha \rfloor$ and $\lfloor n\beta \rfloor$ split $\mathbb{Z}_{\geq 1}$. So do $\lfloor n\alpha \rfloor$, $\lfloor (2n)\beta \rfloor$, $\lfloor (2n-1)\beta \rfloor$.

- Graham 1973: All irrational DCS are compositions of integer DCS.

- So 2 moduli are same for $m \geq 3$. Only the rational case is left open.

• Scheduling: Kubiak, 2003; Brauner, Crama, 2004; Brauner, Jost, 2008.