RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2017

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let G be a group and let $H \subset G$ be a proper subgroup containing all other proper subgroups of G. Show the following:
a) H is normal.
b) G is a cyclic group.
c) G is a finite group.

2. Let g be an invertible $n \times n$ complex matrix. Show that g can be written as

$$g = su = us,$$

where s is diagonalizable and all eigenvalues of u are equal to 1.

3. List, up to isomorphism, all finite abelian groups G such that the order of every element of G divides 55, and the number n_{55} of elements of order exactly 55 satisfies

$$10^2 \le n_{55} \le 10^3.$$

You must prove that your list is accurate.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let G be a nontrivial finite group of prime power order, and let H be a normal subgroup of G. Show that H contains at least one non-identity element of the center of G.

5. Let $GL(n, F)$ denote the group of $n \times n$ invertible matrices with entries in the field F. Prove that $g_1, g_2 \in GL(n, \mathbb{Q})$ are conjugate in $GL(n, \mathbb{Q})$ if and only if they are conjugate in $GL(n, \mathbb{R})$.

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Use the residue theorem to compute the integral

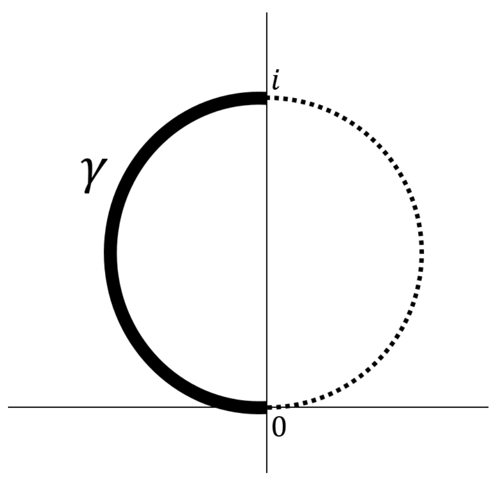
$$\int_{\mathrm{Re}(s)=2} \frac{x^{-s}}{s^3}\,ds \qquad \text{for real } x > 0\,,$$

where the contour is oriented upwards. (Hint: treat the cases of $x < 1$ and $x \geq 1$ separately.)

2. Let S denote the circle of radius $\frac{1}{2}$ centered at $\frac{i}{2} \in \mathbb{C}$ and let

$$\gamma \quad = \quad \{z \in S \mid \mathrm{Re}(z) \leq 0\}$$

be the semicircle shown in the following diagram.



Find a one-to-one and onto holomorphic map from $\mathbb{C} \setminus \gamma$ (the complement of γ in $\mathbb{C}$) to the punctured disk

$$\Delta^* \quad = \quad \{z \in \mathbb{C} \mid 0 < |z| < 1\}.$$

(You may write your answer as a composition of simpler maps, but should justify your steps.)

3. **(A)** Let $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, and let $u(x, y)$ be a harmonic function defined on some open set U containing the closure of B. Prove that

$$u(0,0) \quad = \quad \frac{1}{\pi}\int_B u(x,y)\,dx\,dy\,.$$

(B) Suppose, in addition to the assumptions of part (A), that $\{u_n(x,y)\}_{n=1}^{\infty}$ is a sequence of harmonic functions on U such that

$$\lim_{n\to\infty} \int_B |u_n(x,y) - u(x,y)|\, dx\, dy \;=\; 0\,.$$

Show that $\lim_{n\to\infty} u_n(x,y) = u(x,y)$ for all $(x,y) \in B$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

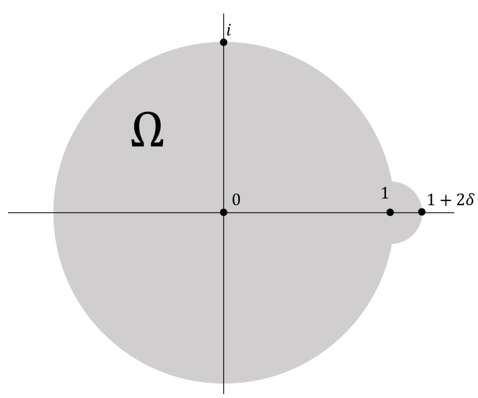
4. Let $f(z)$ be a holomorphic function over the unit disk $\Delta := \{z \in \mathbb{C} \mid |z| < 1\}$. Suppose that $|f(z)| \le 1$ for $z \in \Delta$. Show that for any positive integer n, there is a polynomial $p_n(z)$ of degree at most n such that

$$|f(z) - p_n(z)| \;\le\; (n+2)\,|z|^{n+1}$$

for any $z \in \Delta$.

5. Fix $\delta > 0$ and let

$$\Omega \;=\; \{z \in \mathbb{C} \mid |z| < 1\} \;\cup\; \{z \in \mathbb{C} \mid |z-1| < 2\delta\}\,.$$



Assume that $f(z)$ is a holomorphic function on Ω which has a Taylor series expansion $\sum_{n\ge 0} a_n z^n$ at $z = 0$ such that a_n is a non-negative real number for all $n \ge 0$.

(A) Prove that the derivatives $f^{(k)}(1)$ are real for all $k \geq 0$ and, moreover,

$$f^{(k)}(1) \geq \frac{m!}{(m-k)!} a_m$$

for all $0 \leq k \leq m$.

(B) Prove that $\sum_{n \geq 0} a_n z^n$ has radius of convergence strictly greater than 1.

End of Session 2

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Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

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Part I. Answer all questions.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^1(X, \mathcal{M}, \mu)$ be such that $f(x) > 0$ almost everywhere. Let $E \in \mathcal{M}$ be such that

$$\int_E f \, d\mu < \infty \, .$$

Prove that

$$\lim_{k \to +\infty} \int_E f^{1/k} \mathrm{d}\mu(x) = \mu(E) \, .$$

2. Let X be a set, and let μ^* be an outer measure on X such that $\mu^*(X) < \infty$. Define ν^* by letting

$$\nu^*(E) = \sqrt{\mu^*(E)} \qquad \text{for every} \quad E \subseteq X \, .$$

Prove that

1. ν^* is an outer measure.
2. A subset $A \subseteq X$ belongs to the Carathéodory σ-algebra of ν^* (the σ-algebra of ν^*-measurable sets) if and only if either $\nu^*(A) = 0$ or $\nu^*(A^c) = 0$.

3. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a nonnegative measurable function. Define a function $f : \mathbb{R} \mapsto \mathbb{C}$ by letting

$$f(x) = \int_{-\infty}^{\infty} \frac{e^{ixy}}{1 + y^2 + \sigma(y)} dy \, . \tag{1}$$

Prove that

1. The function f is well defined (that is, the integral of (1) exists) and continuous for all $x \in \mathbb{R}$.
2. If we make the additional assumption that σ is bounded or integrable, then f is differentiable on $\mathbb{R}\setminus\{0\}$ but is not differentiable at 0. You are allowed to use the identity

$$\int_{-\infty}^{\infty} \frac{e^{ixy}}{1 + y^2} dy = \pi e^{-|x|} \, ,$$

valid for all real x.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let (X, d) be a metric space. Prove that if (X, d) is not compact, then there exists an unbounded continuous function $f : X \to \mathbb{R}$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $L^1(X, \mathcal{M}, \mu)$, and let $f \in L^1(X, \mathcal{M}, \mu)$. Suppose that

$$\lim_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu = 0 \, .$$

Suppose also that

$$C := \sup \left\{ \int_X |f_n|^4 \, \mathrm{d}\mu : n \in \mathbb{N} \right\} < \infty \, .$$

Prove that

1. $\int_X |f|^4 \, \mathrm{d}\mu \leq C$.
2. $\int_X |f_n|^2 \, \mathrm{d}\mu < \infty$ for all $n \in \mathbb{N}$, and $\int_X |f|^2 \, \mathrm{d}\mu < \infty$.
3. $\lim_{n\to\infty} \int_X |f_n - f|^2 \, \mathrm{d}\mu = 0$.