This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.
First Day – Part I: Answer each of the following three questions.

1. Let $U$ be an open connected subset of $\mathbb{R}^n$. Show that $U$ is pathwise connected, i.e., that any two points of $U$ may be connected by a continuous path lying entirely in $U$.

2. Find a conformal map taking the disk $\{z \mid |z| < 1/2\}$ to the half-plane $\{z \mid \text{Re}(z) > 1\}$. Justify the correctness of your solution.

3. Let $M$ be an $n \times n$ matrix over $\mathbb{C}$. For $1 \leq i \leq n$, let $a_i = \sum_{j=1}^{n} |M_{ij}|$. Prove that if $\lambda$ is an eigenvalue of $M$, then $|\lambda| \leq \max_{1 \leq i \leq n} a_i$. 
First Day – Part II: Answer three out of the following six questions.

4. Let $f : [0, 1] \to [0, 1]$ be a continuous, nondecreasing function such that $f(0) = 0$ and $f(1) = 1$. For such an $f$, $f'(x)$ exists for almost every $x$ in $[0, 1]$. Use Fatou’s lemma to show that

$$\int_0^1 f'(x) \, dx \leq 1.$$ 

Give an example to show that $\int_0^1 f'(x) \, dx < 1$ is a possibility.

5. The orthogonal group $O_2(\mathbb{R})$ is the set of all $2 \times 2$ matrices $A$ over $\mathbb{R}$ satisfying $AA^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $G$ be a finite subgroup of $O_2(\mathbb{R})$. Show that $G$ is either a cyclic group or a dihedral group.

6. Show that there are at least 3 non-abelian groups of order 16. Make sure to show the groups are different.

7. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers satisfying $a_i + a_j \geq a_{i+j}$ for all $i,j \geq 1$. Prove that $\limsup_{i \to \infty} \frac{a_i}{i} \leq \frac{a_j}{j}$ for all $j \geq 1$. Show that $\{a_i/i : i \geq 1\}$ converges to a finite limit.

8. Let $M$ denote the surface in $\mathbb{R}^3$ defined by the equation $x^2 + y^3 + z^5 = 0$, and let $S^2$ be the sphere of radius 1 about the origin. Show that $M \cap S$ is a curve.

9. Let $f$ be a function which is analytic everywhere on the complex plane except at the points $0, i, -i, \text{ and } \infty$, where it has poles. Show that

$$f(z) = \frac{P(z)}{(z(z^2 + 1))^m}$$

for some polynomial $P$ and integer $m$. 
Second Day – Part I: Answer each of the following three questions.

1. In this problem, measurable means Lebesgue measurable. Let $f$ be a real-valued function such that $f^{-1}([a, \infty))$ is measurable for every real number $a$. Show that $f^{-1}(B)$ is measurable for every Borel subset $B$ of $\mathbb{R}$.

2. Let $U \subset \mathbb{R}^n$ be open and suppose $x_0 \in U$. Let $f$ be a function which is continuous on $U$ and continuously differentiable on $U - \{x_0\}$. If

$$
\ell_i = \lim_{x \to x_0} \frac{\partial f}{\partial x_i}(x)
$$

exists and is finite for $1 \leq i \leq n$, show that $f$ is continuously differentiable on $U$.

3. Let $f(x) = x^5 - 5x^2 + 1$. Prove that $f$ has exactly 3 real roots and is irreducible over the rational numbers $\mathbb{Q}$. 

Second Day – Part II: Answer three of the following six questions.

4. Let $f$ be analytic in a neighborhood of the origin in the complex plane. Suppose that there is a sequence $\{a_n\}$ of distinct real numbers converging to 0 and that $f(a_n)$ is real for each $n$. Show that $f(z) = \overline{f(\bar{z})}$ in a neighborhood of 0.

5. Prove that for every prime $p$, the set of non-zero elements of $\mathbb{Z}/p\mathbb{Z}$ form a cyclic group of order $p - 1$ under multiplication.

6. Let $A$ be a $2 \times 2$ matrix over $\mathbb{C}$. Prove that the number of matrices $X$ satisfying $X^2 = A$ is 0, 2, 4, or $\infty$.

7. Let $X_1, X_2, \ldots$ be a sequence of finite sets, each endowed with the discrete topology, and consider the infinite product $\prod_1^\infty X_i$ as a topological space with the product topology. Suppose that for every $n > 1$, $f_n$ is a function from $X_n$ to $X_{n-1}$. Let $X$ be the subset of $\prod_1^\infty X_i$ consisting of all sequences $(x_1, x_2, \ldots)$ with $f_n(x_n) = x_{n-1}$ for every $n > 1$. Show that $X$ is compact in the subspace topology.

8. Let $A$ be a $3 \times 3$ symmetric matrix over $\mathbb{R}$ with $\det A < 0$ and $a_{11} + a_{22} + a_{33} > 0$. Show that there is a matrix $P$ such that

$$PAP^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function such that $\int_0^\infty |f(x)| \, dx < \infty$. Assume that $\int_a^b f(x) \, dx = 0$ for all $a < b$. Prove that $f(x) = 0$ for almost every $x$. 