Rutgers - Graduate Program in Mathematics

Written Qualifying Examination

Fall 1996

This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.

Typeset by $\mathcal{AMS}$-TEX

First Day – Part I: Answer each of the following three questions.

1. Let U be an open connected subset of $\mathbb{R}^n$. Show that U is pathwise connected, i.e. that any two points of U may be connected by a continuous path lying entirely in U.

2. Find a conformal map taking the disk $\{z \mid \ |z| < 1/2\}$ to the half-plane $\{z \mid \mathrm{Re}(z) > 1\}$. Justify the correctness of your solution.

3. Let M be an $n \times n$ matrix over C. For $1 \le i \le n$, let $a_i = \sum_{j=1}^{n} |M_{ij}|$. Prove that if λ is an eigenvalue of M, then $|\lambda| \le \max_{1 \le i \le n} a_i$.

First Day – Part II: Answer <u>three</u> out of the following six questions.

4. Let $f : [0,1] \to [0,1]$ be a continuous, nondecreasing function such that $f(0) = 0$ and $f(1) = 1$. For such an f, $f'(x)$ exists for almost every x in $[0,1]$. Use Fatou's lemma to show that

$$\int_0^1 f'(x)\,dx \le 1.$$

Give an example to show that $\int_0^1 f'(x)\,dx < 1$ is a possibility.

5. The orthogonal group $O_2(\mathbb{R})$ is the set of all 2×2 matrices A over $\mathbb{R}$ satisfying $AA^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let G be a finite subgroup of $O_2(\mathbb{R})$. Show that G is either a cyclic group or a dihedral group.

6. Show that there are at least 3 non-abelian groups of order 16. Make sure to show the groups are different.

7. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers satisfying $a_i + a_j \ge a_{i+j}$ for all $i, j \ge 1$. Prove that $\limsup_{i\to\infty} \frac{a_i}{i} \le \frac{a_j}{j}$ for all $j \ge 1$. Show that $\{a_i/i : i \ge 1\}$ converges to a finite limit.

8. Let M denote the surface in $\mathbb{R}^3$ defined by the equation $x^2 + y^3 + z^5 = 0$, and let S^2 be the sphere of radius 1 about the origin. Show that $M \cap S$ is a curve.

9. Let f be a function which is analytic everywhere on the complex plane except at the points 0, i, $-i$, and ∞, where it has poles. Show that

$$f(z) = \frac{P(z)}{(z(z^2+1))^m}$$

for some polynomial P and integer m.

Second Day – Part I: Answer each of the following three questions.

1. In this problem, *measurable* means *Lebesgue measurable*. Let f be a real-valued function such that $f^{-1}([a,\infty))$ is measurable for every real number a. Show that $f^{-1}(B)$ is measurable for every Borel subset B of $\mathbb{R}$.

2. Let $U \subset \mathbb{R}^n$ be open and suppose $x_0 \in U$. Let f be a function which is continuous on U and continuously differentiable on $U - \{x_0\}$. If

$$\ell_i = \lim_{x \to x_0} \frac{\partial f}{\partial x_i}(x)$$

exists and is finite for $1 \leq i \leq n$, show that f is continuously differentiable on U.

3. Let $f(x) = x^5 - 5x^2 + 1$. Prove that f has exactly 3 real roots and is irreducible over the rational numbers $\mathbb{Q}$.

Second Day – Part II: Answer <u>three</u> of the following six questions.

4. Let f be analytic in a neighborhood of the origin in the complex plane. Suppose that there is a sequence $\{a_n\}$ of distinct real numbers converging to 0 and that $f(a_n)$ is real for each n. Show that $f(z) = \overline{f(\bar{z})}$ in a neighborhood of 0.

5. Prove that for every prime p, the set of non-zero elements of $\mathbb{Z}/p\mathbb{Z}$ form a cyclic group of order $p-1$ under multiplication.

6. Let A be a 2×2 matrix over C. Prove that the number of matrices X satisfying $X^2 = A$ is 0, 2, 4, or ∞.

7. Let $X_1, X_2, \ldots$ be a sequence of finite sets, each endowed with the discrete topology, and consider the infinite product $\prod_1^\infty X_i$ as a topological space with the product topology. Suppose that for every $n > 1$, f_n is a function from X_n to X_{n-1}. Let X be the subset of $\prod_1^\infty X_i$ consisting of all sequences $(x_1, x_2, \ldots)$ with $f_n(x_n) = x_{n-1}$ for every $n > 1$. Show that X is compact in the subspace topology.

8. Let A be a 3×3 symmetric matrix over $\mathbb{R}$ with $\det A < 0$ and $a_{11} + a_{22} + a_{33} > 0$. Show that there is a matrix P such that

$$PAP^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function such that $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$. Assume that $\displaystyle\int_a^b f(x)\, dx = 0$ for all $a < b$. Prove that $f(x) = 0$ for almost every x.