RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Spring 2000, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Let p and q be prime numbers, such that $p < q$ but p does not divide $q - 1$. **Prove** that any group of order pq is commutative.

2. Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ be subsets of $[0, 1]$. We do not assume that these sets are measurable. **Prove** that

$$\lim_{n\to\infty} m^*(A_n) = m^*\left(\bigcup_{n=1}^{\infty} A_n\right)$$

where m^* denotes Lebesgue outer measure.

3. **Prove** that the formula $f(z) = \sum_{n=1}^{\infty} \left[1 - \cos\left(\frac{z}{n}\right)\right]$ defines an entire function (*i.e.*, a function defined and holomorphic on the entire complex plane $\mathbb{C}$.

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Suppose that T is a square complex matrix whose minimal polynomial is $\mu_T(t) = t^3 - 1$ and whose characteristic polynomial is $\chi_T(t) = (t^3 - 1)^2$. Suppose that S is a complex matrix with the same number of rows and columns as T, such that the only (complex) eigenvalue of S is 0 and such that $ST = TS$. **Prove** that $S^2 = 0$.

5. Let G be a group which is also a topological space, and assume that for each $g \in G$ the translation mapping $\lambda_g(h) = gh$, is continuous. **Prove** that any subgroup of G which is open is necessarily closed, but **give an example** to show that the converse is false in general.

6. Let $\mathbb{R}$ and $\mathbb{R}^+$ denote the real numbers and the nonnegative real numbers respectively, as usual. Let γ and k be real constants. **Solve** the initial-value problem for a function $x : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$\begin{aligned} \frac{d^2x(t)}{dt^2} + 2\gamma \frac{dx(t)}{dt} + k^2 x(t) &= 0 \quad \text{for } t > 0, \text{ and} \\ x(0) &= 1 \\ x'(0) &= 0 \end{aligned}$$

by using the Laplace-transform technique, indicating the way in which you employ this technique.

7. (a) Consider the space $L^2[0,1] = \left\{ f : \int_0^1 |f(x)|^2 \, dx < \infty \right\}$, and consider the functions $f_r(x) = e^{irx}$ (where $r \in \mathbb{R}^+$) as elements of this space. **Prove** that f_r converges weakly to zero in $L^2[0,1]$ as $r \to \infty$.

(b) Now consider the space $L^2([0,1], x\,dx) = \left\{ f : \int_0^1 |f(x)|^2 x \, dx < \infty \right\}$; **prove** that f_r converges weakly to zero as $r \to \infty$ in this space also.

8. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a continuous function that is holomorphic and nonconstant on $\mathbb{D}$ and such that $|f(z)| < 1$ for $|z| = 1$. **Prove** that there is one and only one fixed point of the function f in $\mathbb{D}$, *i.e.*, one and only one point z_0 for which $f(z_0) = z_0$.

9. Let $\{f_n : [0,1] \to [0,\infty)\}_{n=1}^{\infty}$ be a sequence of continuous functions and let $f : [0,1] \to [0,\infty)$ be a function for which (a) and (b) below hold:

(a) If $m \neq n$ and $x \in [0,1]$ then $f_m(x) \neq f_n(x)$;

(b) $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ for (Lebesgue-)almost all $x \in [0,1]$.

Prove that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Spring 2000, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. **Given an example** of a sequence $\{f_n : [0,1] \to \mathbb{R}\}_{n=1}^{\infty}$ of Lebesgue-integrable functions, such that $f_n \to 0$ in L^1 norm but $\lim_{n\to\infty} f_n(x)$ does not exist for any $x \in [0,1]$.

2. Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let $T : V \to V$ be a linear transformation. Make V into a module for the polynomial ring $\mathbb{F}[t]$ by defining $tv = Tv$ for all $v \in V$. Assume that

$$V \cong \frac{\mathbb{F}[t]}{(f_1)} \oplus \frac{\mathbb{F}[t]}{(f_2)} \oplus \cdots \oplus \frac{\mathbb{F}[t]}{(f_m)} \quad \text{as } \mathbb{F}[t] - \text{modules}$$

where $f_1, f_2, \ldots, f_m \in \mathbb{F}[t]$. No divisibility assumption is made about $f_1, f_2, \ldots, f_m$. **Express** the following in terms of $f_1, f_2, \ldots, f_m$, and **prove** that your answers are correct:
(a) $\dim V$;
(b) The minimal polynomial $\mu_T(t)$ of T;
(c) The characteristic polynomial $\chi_T(t)$ of T.

3. Let $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$, where $0 < a < 1$ and b is an odd natural number. **Prove** that $f(x)$ is continuous for all $x \in \mathbb{R}$.

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. **Prove** that the limit $\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \log n\right]$ exists and is positive. For full credit, also **prove** that $\sum_{k=1}^{n}\frac{1}{k} = \gamma + \log n + O\left(\frac{1}{n}\right)$, where $O\left(\frac{1}{n}\right)$ denotes a "remainder term" whose quotient by $1/n$ remains bounded as $n \to \infty$.

5. **Find** the (real) value of each of the following integrals (in (b), for all values of the real parameter k):

(5a)
$$\int_0^\infty \frac{dx}{1+x^3}\ ;$$

(5a)
$$\int_0^\infty \frac{\sin(kx)}{x}dx\ .$$

6. Let A be an invertible complex matrix. **Prove** that for any natural number k there exists a matrix X such that $X^k = A$.

7. (a) **Exhibit** a power series $\sum_{n=0}^{\infty} a_n z^n$ that has all of the following three properties: (i) the series converges for all complex $|z| < 1$; (ii) $\sum_{n=0}^{\infty} |a_n|$ diverges; (iii) $\sum_{n=0}^{\infty} |a_n|^2$ converges.

(b) **Prove** that for any sequence of coefficients such that $\sum_{n=0}^{\infty} |a_n|^2$ converges (to a finite value), the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ must converge for all (complex) $|z| < 1$ and the function it defines must satisfy $\lim_{|z|\to 1^-} \sqrt{1-|z|}\cdot f(z) = 0$.

8. (a) If $K \subset \mathbb{R}^n$ is a closed, bounded, nonempty set, **define** what it means for K to be connected.

(b) Suppose that $\{K_i\}_{i=1}^{\infty}$ is a sequence of closed, bounded, nonempty, connected sets in $\mathbb{R}^n$ that is decreasing, *i.e.*, satisfies $K_1 \supseteq K_2 \supseteq \cdot$. **Prove** that $K = \bigcap_{i=1}^{\infty} K_i$ is also nonempty and connected. (You may use well-known fundamental properties of $\mathbb{R}$ and $\mathbb{R}^n$, but you must **indicate explicitly** what properties you use.)

9. Say that a group G is *revolutionary* if and only if there exist a natural number n and a chain of subgroups

$$\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots G_n = G\,,$$

each normal in the next, such that the order of G_i/G_{i-1} divides either $1848 = 2^3 \cdot 3 \cdot 7 \cdot 11$ or $1917 = 3^3 \cdot 71$ for each $i = 1, \dots, n$. (Here e is the identity element of G.)

(a) **Prove** that if N is a normal subgroup of a group G such that both N and G/N are revolutionary, then G is revolutionary.

(b) **Prove** that every subgroup of a revolutionary group is revolutionary.