RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Fall 2000, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Let f_n be a sequence of analytic functions on the open unit disc $\mathbf{D}$, and continuous on the closed unit disc. Assume furhter that **each** f_n is injective and that f_n converges to an analytic function f uniformly on compact subsets of $\mathbf{D}$. Prove that either f is injective, or that f is identically a constant.
Note: It is not enough to quote a theorem here, a full argument is needed.

2. Let $f \in L^\infty[0,1]$. Define, for $0 < p < \infty$,

$$F(p) = (\int_0^1 |f|^p dx)^{1/p}.$$

Prove:
(a) $F(p)$ is an increasing function of p.
(b)

$$\lim_{p \to \infty} F(p) = \|f\|_{L^\infty[0,1]}$$

3. Show that the polynomial $x^5 - x + 1$ is not the product of two polynomials with integral coefficient and positive degree.

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Give an example of a real-valued sequence $\{s_n\}$, such that s_n is **bounded**, divergent, and

$$\lim_{n\to\infty} (s_{n+1} - s_n) = 0.$$

5. Let f be continuous in $[0, \infty)$ and $f \geq 0$. Suppose for all $x \in [0, \infty)$,

$$f(x) \leq \int_0^x f(t)dt.$$

Prove, $f \equiv 0$.

6. Let $f(z)$ be a meromorphic function on the open unit disc $\mathbf{D}$. Assume that f has a continuous extension to the closed unit disk and is finite complex-valued on $|z| = 1$. If in addition $|f(z)| = 1$ on the boundary $|z| = 1$ of $\mathbf{D}$, prove that f is the restriction to $\mathbf{D}$ of a rational function.

7. Let f be an entire, analytic function that is conformal. Prove, that

$$h(x, y) = \ln |f'(x + iy)|,$$

is a harmonic function.

8. Let $GL(2, F)$ be the group of 2×2 invertible matrices with entries in a finite field F with $q = p^n$ elements, with p a prime. Find the order of this group. Show that every $p-$Sylow subgroup of $GL(2, F)$ is isomorphic to the additive group of F.

9. Consider the vector space $M(n, \mathbf{C})$, whose elements are $n \times n$ matrices with entries which are complex numbers. Let $A \in M(n, \mathbf{C})$, have the property that **each** row and **each** column of A has precisely one entry equal to 1 and the rest of the entries are zero. Prove that A is similar to a diagonal matrix.

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Fall 2000, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Using complex analysis, evaluate the integral,

$$\int_0^\infty \frac{1}{(1+x^5)}dx.$$

Do the complex algebra and display the answer as an explicit real number.

2. Let $\{a_n\}$ be a bounded sequence of real numbers. Let A be the set

$$A = \{x : \text{there are infinitely many } n \text{ so that } a_n < x\}.$$

Prove that,

$$\inf A = \liminf a_n.$$

3. Let M be the subgroup of the group L of ordered triples of integers under addition, generated by $(1, 2, 3)$,$(2, 3, 1)$,$(3, 1, 2)$. Using an appropriate theorem on the structure of the subgroups of L, answer the problems below.
(a) Compute the index of M in L.
(b) Prove that M is not a direct summand of L as modules over the integers.
(c) Is the subgroup generated by $(1, 2, 3)$,$(2, 3, 1)$ a direct summand of L?

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let f be a measurable function on $[0,1]$ with $f \geq 0$ a.e. Prove,

$$\int_0^\infty e^{-\lambda} |\{x \in [0,1] : \lambda > f(x)\}| \, d\lambda = \int_0^1 e^{-f(x)} dx.$$

5. Let f_n be a sequence of C^1 functions on the closed interval $[0,1]$. Assume there is a constant C, **independent** of n, such that,

$$\int_0^1 (|f_n|^2 + |f_n'|^2) dx \leq C^2.$$

(a) Prove that there exist points $x_n \in [0,1]$, such that

$$|f_n(x_n)| \leq C.$$

(b) Prove for **any** given $L^p[0,1]$ space, there exists a subsequence f_{n_k} and a function $f \in L^p[0,1]$ such that,

$$\int_0^1 |f - f_{n_k}|^p dx \to 0, \ k \to infty.$$

You may want to use part (a) to prove (b).

6.(a) Construct an analytic $1-1$ map f from the open unit disc $\mathbf{D}$ **onto** the upper half plane $\mathbf{H} = \{x + iy : y > 0\}$, such that $f(0) = i$.
(b) Using part (a) or by other means, construct an analytic $1-1$ map g, from the open unit disc $\mathbf{D}$ **onto** the **horizontal** strip D,

$$S = \{x + iy : -\pi/2 < y < \pi/2\},$$

such that $g(0) = 0$.

7. Let $u : \mathbf{R}^2 \to \mathbf{R}^2$ be an entire harmonic function with Taylor series

$$u(x_1, x_2) = \sum_{m=0}^\infty \sum_{n=0}^\infty a_{n,m} x_1^m x_2^n$$

(Note that the radii of convergence in x_1 and x_2 are infinite.) Let $U : \mathbf{C}^2 \to \mathbf{C}$ be the extension of u to $\mathbf{C}^2$ obtained by replacing x_1 by $z_1 = x_1 + iy_1$ and x_2 by $z_2 = x_2 + iy_2$ in the Taylor expansion of u.
Let $z = x + iy$. Show that $f : \mathbf{C} \to \mathbf{C}$ given by

$$f(z) \equiv 2U(z/w, -iz/2) - u(0,0)$$

is an entire analytic function on $\mathbf{C}$ whose real part is $u(x, y)$.

8. Let V be a finite diemnsional vector space over the field of rational numbers. Let $T : V \to V$ be a linear transformation such that T^5 is the identity transformation. Show either, the dimension of V is a multiple of 4, or there exists a non-zero vector $v \in V$ such that T fixes v. Here either/or means both situations could occur, the situations are not mutually exclusive.

9. Let $SO(3)$ be the group of proper rotations of $\mathbf{R}^3$, that is the 3×3 orthogonal matrices with determinant 1. Let A be the subgroup of rotations about the $x-$axis, B the subgroup of rotations about the $y-$axis.
(a) Prove that there exist $g \in SO(3)$ that cannot be expressed as ab, where $a \in A$ and $b \in B$.
(b) Is $SO(3)$ generated by the subgroups A and B?