RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Fall 2001, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Suppose that R is a unique factorization domain and d is a nonzero element of R. Prove that there are only a finite number of distinct principal ideals that contain the ideal (d). Here (d) denotes the principal ideal generated by d.

2. Let A be a subset of $[0,1]$. Let B be the complement of A in $[0,1]$. Assume that $m^*(A) + m^*(B) = 1$, where m^* is Lebesgue outer measure. Prove that A is Lebesgue measurable.

3. If A is a countable subset of the plane $\mathbb{R}^2$, show that any two points in $\mathbb{R}^2 \setminus A$ are joined by some continuous path $\gamma\colon [0,1] \to \mathbb{R}^2 \setminus A$. That is, given p and q in $\mathbb{R}^2 \setminus A$, show that there is a continuous path $\gamma\colon [0,1] \to \mathbb{R}^2 \setminus A$ joining p and q.

   Hint How many lines are there passing through a given point in the plane?

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. a) Let $f(z)$ be an entire function whose restriction to $\mathbb{R}$ is periodic with period c: that is, $f(x+c) = f(x)$ for all $x \in \mathbb{R}$. Prove that $f(z+c) = f(z)$ for all $z \in \mathbb{C}$.

   b) Let $f(z)$ be an entire function which has two periods $a, b \in \mathbb{C}$ with $\frac{a}{b} \notin \mathbb{R}$. Prove that f is constant.

5. Suppose that V is a real vector space of finite dimension n and $T: V \to V$ is a linear transformation with no repeated eigenvalues. Show that there exists a vector $v \in V$ such that $\{v, Tv, T^2v, \ldots, T^{n-1}v\}$ is a basis of V.

6. a) Suppose that $\alpha \in \mathbb{R}$ and $\alpha < 1$. Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \int_0^\infty t^{x^2-\alpha} e^{-t}\, dt\,.$$

   Prove that f is continuous.

   b) Decide if this result is also true when $\alpha = 1$ and prove your answer.

7. Suppose that $f: [0,1] \to \mathbb{R}$ is a continuously differentiable function. Prove that there exists a sequence $\{p_j\}_{j=1}^\infty$ of polynomial functions on $[0,1]$ so that $p_j \to f$ and $p_j' \to f'$ uniformly as $j \to \infty$.

8. Suppose that (X, d) is a complete metric space which is *bounded*, in the sense that there exists a real number B such that

$$d(x,y) \le B \text{ for all } x \in X \text{ and } y \in X\,.$$

   Is it true that every continuous function $f: X \to \mathbb{R}$ is bounded? Prove or give a counterexample.

9. Let $f \in \mathbb{Q}[t]$ and suppose that $f^n \in \mathbb{Z}[t]$ for some positive integer n. Prove that $f \in \mathbb{Z}[t]$.
   Hint Use Gauss's Lemma.

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Fall 2001, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. a) An element a of a ring is *nilpotent* if $a^n = 0$ for some positive integer n. Prove that in a commutative ring, if a and b are nilpotent, then $a + b$ is nilpotent.

   b) Show that this result is false in general if the ring is not commutative.

2. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 4x + 5)^2}.$$

   Use the method of residues and explain why it applies.

3. Suppose that S is a subset of $[0, 1]$ such that the complement $[0, 1] \setminus S$ is finite. If $f\colon [0, 1] \to \mathbb{R}$ is continuous and $f'(x)$ exists and is nonnegative for all $x \in S$ (that is, $f'(x) \geq 0$ for $x \in S$), then prove that $f(1) \geq f(0)$.

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. a) Prove the theorem that every eigenvalue of a Hermitian matrix is real.

   b) Prove that every eigenvalue λ of a skew-symmetric real matrix has the form $\lambda = it$ for some $t \in \mathbb{R}$.

   Hint You may use a).

5. Let G be a group which acts transitively on finite sets Ω and Ψ, with $1 < |\Omega| < |\Psi|$. If G is simple, show that $|\Psi|$ cannot be prime.

6. Give an example of a continuous function $f\colon [0,1] \to \mathbb{R}$ and a Lebesgue measurable set $A \subset \mathbb{R}$ such that $f^{-1}(A)$ is not Lebesgue measurable.

7. Suppose that f is a meromorphic function on $\mathbb{C}$. Assume that f has no pole at 0 and that f satisfies

$$f(z+1) = f(z) = f(z+i)$$

   for all z. Assume also that f is not constant. Let R be the radius of convergence of the Maclaurin series of f (that is, the Taylor series of f centered at 0). Prove that $R \leq \dfrac{1}{\sqrt{2}}$.

8. Suppose that $A(t)$ is a continuously differentiable square-matrix-valued function of a real variable t: that is, for each ordered pair of integers, (i,j) with $1 \leq i, j \leq n$, the $(i,j)^{\text{th}}$ entry $a_{ij}(t)$ of $A(t)$ is a continuously differentiable function of t. Suppose also that λ_0 is an eigenvalue of $A(0)$ with multiplicity 1, so that λ_0 is a simple root of the characteristic polynomial of $A(0)$. Show that when t is small, $A(t)$ has an eigenvalue $\lambda(t)$ which depends differentiably on t, and which satisfies $\lambda(0) = \lambda_0$.

9. Suppose that $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$ is a monic polynomial with complex coefficients. If $r \in \mathbb{C}$ and $p(r) = 0$, show that

$$|r| \leq \max(1, |a_0| + |a_1| + \cdots + |a_{n-1}|)\,.$$