RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination

January 2005, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Prove that the set $\mathbb{Q}$ of all rational numbers is not a G_δ-subset of the real line.

2. Suppose that a is a real number satisfying $a > 1$. Use the Residue Theorem to compute

$$\int_0^{2\pi} \frac{1}{a + \cos\theta}\, d\theta.$$

3. Let A be an $n \times n$ symmetric real matrix with n distinct eigenvalues. Prove that every $n \times n$ real matrix commuting with A is symmetric.

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First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Suppose U is an open subset of $\mathbb{C}$. If f is a holomorphic function in U, then a holomorphic function g is called a *holomorphic logarithm* of f if $e^g = f$ in U.

a) Prove that if U is simply connected, then a holomorphic function f which is never 0 always has a holomorphic logarithm, g.

b) Suppose that a holomorphic function f has a holomorphic logarithm g in a connected open subset U of $\mathbb{C}$. Describe with explanation *all* holomorphic logarithms of f in U.

5. Let p be a prime number. Prove that the polynomial

$$f(x) = x^{p-1} + x^{p-2} + \cdots + 1$$

is irreducible over the ring of rational numbers.

6. Suppose $P(z) = z^{17} + z^{66} - 5z^4 + 2$.

a) How many zeros (counted with multiplicity) does $P(z)$ have in the unit disk $D_1 = \{z \in \mathbb{C} : |z| < 1\}$?

b) How many zeros (counted with multiplicity) does $P(z)$ have in the disk $D_2 = \{z \in \mathbb{C} : |z| < 2\}$?

7. Let (X, d) be a metric space. Suppose that for any $\epsilon > 0$ there exists a finite set $Y_\epsilon \subset X$ such that for any $x \in X$

$$\min_{y \in Y_\epsilon} d(x, y) \leq \epsilon.$$

Prove that the completion of X is compact.

8. Prove that no group of order 30 is simple.

9. Let $f : [0, 1] \to \mathbb{R}$ be a function of bounded variation. Prove or construct counter-examples to the following statements:

a) If $\phi : \mathbb{R} \to \mathbb{R}$ is continuous then the composition $\phi(f)$ is of bounded variation on $[0, 1]$.

b) If $\phi : \mathbb{R} \to \mathbb{R}$ is of bounded variation on $\mathbb{R}$ then the composition $\phi(f)$ is of bounded variation on $[0, 1]$.

10. Let $E_1, E_2, \ldots, E_n$ be measurable subsets of a segment $[a, b]$. Suppose that any point x belongs to at least q of those sets. Prove that at least one of the sets $E_1, E_2, \ldots, E_n$ has measure greater or equal to $(b - a)\frac{q}{n}$.

11. Let R be a commutative ring with unit and $f(x), g(x) \in R[x]$. Suppose that the product of all coefficients in $f(x)$ is not zero and that the product $f(x)g(x) = 0$. Prove that there exists a nonzero element $a \in R$ such that $ag(x) = 0$.

12. Let Q denote the first quarter of the unit disk, that is, $Q = \{z \in \mathbb{C} : \Re z > 0, \Im z > 0, 0 < |z| < 1\}$. Find a conformal mapping from Q to the unit disc $D_1 = \{z \in \mathbb{C} : |z| < 1\}$. Draw pictures of the domain and range of all mappings you may use to describe your answer.

Day 1 Exam End

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Written Qualifying Examination

January 2005, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Let $E \subset \mathbb{R}$ be a non-measurable set, and $A \subset \mathbb{R}$ a set of measure zero. Prove that the set $E \cap (\mathbb{R} \setminus A)$ is not measurable.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that there is a constant $R > 0$ such that $|z| > R$ implies $|f(z)| > R$. Show that $f(z)$ is a polynomial.

3. Prove that $\mathbb{Z} \times 3\mathbb{Z}$ is a maximal ideal in $\mathbb{Z} \times \mathbb{Z}$.

Exam continues on next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Suppose that $h(x, y)$ is a harmonic function on $\mathbb{R}^2$, that $h(x, y)$ is *never* zero and that $h(-1, 2) = 1766$. What is $h(x, y)$?

5. Suppose that $B = \{b_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that

$$\sum_{n=1}^{\infty} \frac{1}{|b_n|^3}$$

converges and $|b_n| < |b_{n+1}|$. If $f(z)$ is defined by

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{1}{z - b_n} + \frac{1}{b_n} + \frac{z}{b_n^2} \right),$$

prove that $f(z)$ is holomorphic in $\mathbb{C} \setminus B$.

6. Suppose that the symmetric group Σ_4 acts transitively on a finite set X having 8 elements. How many different subgroups of Σ_4 can occur as stabilizers of points of X?

7. Define $f : [0, 1] \to \mathbb{R}$ as follows:
If $x \in [0, 1]$ has decimal expansion $x = 0.n_1 n_2 n_3 \ldots$ then

$$f(x) = \max_i n_i.$$

Prove that the function $f(x)$ is measurable.

8. Let $f \in L^1(\mathbb{R})$. Prove that

$$\int_{\mathbb{R}} |f(x + \epsilon) - f(x)| dx$$

goes to 0 as ϵ goes to 0.

9. Let μ be the Lebesgue measure on $\mathbb{R}$. Prove that there is no subset A on the real axis such that A is Lebesgue measurable and for any interval $I \subset \mathbb{R}$

$$\mu(A \cap I) = \frac{1}{2}\mu(I).$$

10. Consider the two one-dimensional subspaces of $\mathbb{C}^2$, $L_1 = \{(z_1, 0) : z_1 \in \mathbb{C}\}$ and $L_2 = \{(0, z_2) : z_2 \in \mathbb{C}\}$. Show that the complement $\mathbb{C}^2 \setminus (L_1 \cup L_2)$ is connected.

11. Let R be a noncommutative ring with identity, and let $M_n(R)$ be the ring of all matrices over R. If C denotes the center of R and $I \in M_n(R)$ is the identity matrix, show that $C \cdot I = \{\text{diag}(c, \ldots, c)\}$ is the center of $M_n(R)$.

12. Let $a + bi$ be a prime in the Gaussian integers $\mathbb{Z}[i]$ with $a, b \neq 0$. Prove that the integer $a^2 + b^2$ is prime in $\mathbb{Z}$.

Exam Day 2 End