RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 30, 2010, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

- Be sure your ID is on each book that you are submitting
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

First Day—Part I: Answer each of the following three questions

1. Prove that no group of order 3400 is simple.

2. Construct a function $f : \mathbb{R} \to \mathbb{R}$ such

$$\int_{-\infty}^{\infty} |f(x)|^p dx < \infty$$

if and only if $p \in (2, 3)$.

3. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be *of class* C^1 if the partial derivatives $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ exist at every point $(x, y) \in \mathbb{R}^2$, and are continuous functions of (x, y).

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function of class C^1 and that there exists a constant $\alpha > 0$ such that $\frac{\partial f}{\partial y}(x, y) \geq \alpha$ for all (x, y).

(a) Without invoking the implicit function theorem, show that there is a unique continuous function $g(x)$, defined for all $x \in \mathbb{R}$, that solves $f(x, g(x)) = 0$.

(b) Using the mean value theorem, show that $g(x)$ is continuously differentiable.

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Prove that if f is a holomorphic function on $\mathbb{C}\backslash\{0\}$ (the complex plane with the origin removed) and $f(z)$ is real whenever z is real and positive, then $f(z)$ is real whenever z is real and negative.

5. Find a uniformly bounded sequence $\{f_n : \mathbb{R} \to \mathbb{R}\}_{n=1}^{\infty}$ of Lebesgue integrable functions which converges pointwise to 0 but not in the norm of $L^1(\mathbb{R})$.

6. Let $f_n : [0,1] \to [0,\infty)$ be nonnegative measurable functions and suppose $f_n \to f$ (Lebesgue) almost everywhere. Assume also that

$$\lim_{n\to\infty} \int_{[0,1]} f_n \, dx = \int_{[0,1]} f \, dx < \infty.$$

Prove that for any measurable subset A of $[0,1]$,

$$\lim_{n\to\infty} \int_A f_n \, dx = \int_A f \, dx.$$

7. Let U be an open subset of the complex plane $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be a one-to-one holomorphic function. Then the area of the set $f(U)$ is given by one of the following three formulas:

$$\begin{aligned}
\text{Area}(f(U)) &= \int_U |f'(x+iy)| \, dx \, dy \,, \\
\text{Area}(f(U)) &= \int_U |f'(x+iy)|^2 \, dx \, dy \,, \\
\text{Area}(f(U)) &= \int_U \sqrt{1 + |f'(x+iy)|^2} \, dx \, dy \,.
\end{aligned}$$

Determine which one is the correct formula, and prove it. Indicate clearly any theorems from advanced calculus that you use in your proof.

8. A *boolean* ring R is a ring with identity such that $x^2 = x$ for all $x \in R$.

(a) Prove that a boolean ring is commutative and is naturally a vector space over the field $\mathbb{Z}/2\mathbb{Z}$.

(b) Prove that a finite boolean ring is isomorphic to a direct product of copies of $\mathbb{Z}/2\mathbb{Z}$.

(Recall that the direct product of the rings $R_1, ..., R_k$ is $\{(r_1, ..., r_k)|r_i \in R_i\}$ with addition defined by $(r_1, ..., r_k) + (s_1, ..., s_k) = (r_1 + s_1, ..., r_k + s_k)$ and multiplication defined by $(r_1, ..., r_k)(s_1, ..., s_k) = (r_1 s_1, ..., r_k s_k)$.)

9. Let F be a field and let $M_n(F)$ denote the set of all $n \times n$ matrices with entries in F. Then if $A \in M_n(F)$ and rank $A = r$, determine the rank of the linear transformation $T : M_n(F) \to M_n(F)$ given by

$$T(B) = AB$$

and prove your assertion.

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 31, 2010, Day 2

This examination is given in two three-hour sessions, today's being the second part.
At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

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Second Day—Part I: Answer each of the following three questions

1. Show that the rings $\mathbb{Z}[i]/(1+2i)\mathbb{Z}[i]$ and $\mathbb{Z}/5\mathbb{Z}$ are isomorphic.

2. Let $f_0(x)$ be a continuous function on $[0, a]$ for some positive number a. Define

$$f_n(x) = \int_0^x f_{n-1}(t)dt$$

for $n \in \mathbb{N}$ (the set of positive integers) and $x \in [0, a]$. Show that the sequence $\{f_n(x)\}$ is uniformly convergent to 0 on $[0, a]$.

3. Let f be a nonconstant meromorphic function on the complex plane such that $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for every $z \in \mathbb{C}$.

(a) Prove that f must have poles.

(b) Prove that in the square $\{z \in \mathbb{C} \ : \ 0 \leq \operatorname{Re} z < 1, 0 \leq \operatorname{Im} z < 1\}$ the function f must have either

- two (or more) different poles, or else
- one (or more) single pole of order ≥ 2.

The exam continues on the next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let m^* be Lebesgue outer measure on $\mathbb{R}$. If A is a subset of $\mathbb{R}$, let A^c denote its complement.

(a) In terms of m^*, define what it means for $A \subset R$ to be Lebesgue measurable.

(b) Suppose A is not Lebesgue measurable and suppose that B is a set of Lebesgue measure 0. Show that $A \cap B^c$ is also not Lebesgue measurable.

5. Let $\mathbb{N}$ be the set of positive integers and $S : \mathbb{N} \to \mathbb{R}$ a subadditive map, i.e. $S(n_1 + n_2) \leq S(n_1) + S(n_2)$ for $n_1, n_2 \in \mathbb{N}$. Assume that

$$s_0 = \inf_{n \in \mathbb{N}} \frac{S(n)}{n} > 0$$

exists. Prove that

$$\lim_{n \to \infty} \frac{S(n)}{n} = s_0$$

HINT: Use the fact that for any $\epsilon > 0$ you can find some $N_\epsilon \in \mathbb{N}$ such that

$$\frac{1}{N_\epsilon} S(N_\epsilon) \leq s_0 + \epsilon$$

and notice that for any $n > N_\epsilon$ you have

$$n = M N_\epsilon + L$$

with $M \in \mathbb{N}$ and $L \in \mathbb{N}$, and $L < N_\epsilon$.

6. Let

$$I(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{1 + x^2}\, dx$$

for $t \in \mathbb{R}$.

(a) Use general theorems on Lebesgue integrals to prove that $I(t)$ is a continuous function of t. Does your argument also suffice to prove that $I(t)$ is differentiable?

(b) Use complex contour integration to compute $I(t)$ explicitly in terms of elementary functions. Justify any limiting arguments that you employ.

(c) Show that your explicit formula for $I(t)$ is consistent with your conclusions from part (a) regarding continuity and differentiablility.

7. If A is an $n \times n$ matrix over a field F whose minimal polynomial has n distinct roots, prove that if B is an $n \times n$ matrix over F with $AB = BA$, then $B = p(A)$, for some polynomial $p(x) \in F[x]$.

8. Let $G = GL_n(\mathbb{R})$, the group of all invertible $n \times n$ matrices over the real numbers, operate on the set $S = \mathbb{R}^n$ of n-dimensional row vectors by right multiplication.

(a) Describe the decomposition of S into orbits for this operation.

(b) Describe in matrix form the stabilizer of e_1. (e_1 is the vector $(1, 0, \ldots, 0)$.)

(c) Let $v = (v_1, ..., v_n) \in R^n$ be a non-zero vector. Use your answer to (b) to describe in matrix form the stabilizer of v.

9. Prove that if a sequence $\{f_n\}_{n=1}^{\infty}$ of holomorphic functions on a connected open subset Ω of $\mathbb{C}$ converges uniformly on compact sets to a function f, and all the f_n are one-to-one, then either f is one-to-one or f is a constant.

Exam Day 2 End