GRADUATE PROGRAM IN MATHEMATICS
RUTGERS UNIVERSITY

Written Qualifying Examination

Day 1, January, 2012

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3), and answer three of the six questions in Part II (numbered 4–9).

Before handing in your exam,

- Be sure your ID is on each book you are submitting.
- Label the books at the top as Book 1 of X, Book 2 of X, etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

If you work on more than three questions in Part II, indicate clearly which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day–Part I: Answer each of the following three questions.

1. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{4+\sin\theta}$.

2. Suppose that G is a group of order 924. Prove that G has an element of order 77.

3. Suppose that f is Lebesgue integrable on the interval $(0,a)$, where $0 < a < \infty$. For all $0 < x < a$, define

$$g(x) = \int_x^a f \, \mathrm{d}m$$

where m denotes Lebesgue measure. Show that g is integrable on $(0,a)$ and

$$\int_0^a g \, \mathrm{d}m = \int_0^a x f(x) \, \mathrm{d}m.$$

The exam continues on the next page

First Day–Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let U be an open subset of $\mathbb{C}$, and let $\{u_n\}_{n=1}^{\infty}$ be a sequence of harmonic functions such that $|u_n(p)| \leq 1$ for all $p \in U$ and all n. Assume that the limit

$$u(p) = \lim_{n\to\infty} u_n(p)$$

exists for every $p \in U$. Prove that u is a harmonic function on U.

5. Let $T_i : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations, $i = 1, \ldots, n$. Assume that $T_i^n = 0$ for all $i = 1, \ldots, n$, and $T_iT_j = T_jT_i$ for all $i, j = 1, \ldots, n$. Prove that $T_1T_2 \cdots T_n = 0$.

6. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. For any two complex valued measurable functions f and g on X, define

$$d(f, g) := \int_X \frac{|f-g|}{1+|f-g|} d\mu.$$

Show that d is a metric on the space of measurable functions on X if, as usual, we identify functions that are equal almost everywhere. Show also that if $\{f_n\}$ is a sequence of measurable functions on X, and f is measurable on X, then $f_n \longrightarrow f$ in measure if and only if $\lim_{n\to\infty} d(f_n, f) = 0$.

7. Let $V = \mathbb{R}^4$ and let q be the quadratic form $q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 - x_4^2$ on V. Find all the subspaces W of V such that $q|_W \equiv 0$ and W is maximal subject to this condition.

The exam continues on the next page

8. Exhibit a biholomorphic mapping (i.e., a holomorphic mapping with a holomorphic inverse) from the open half-disc $S = \{z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im} z > 0\}$ onto the open unit disc $D = \{z : |z| < 1\}$.

9. Let X be a compact topological space. Let A and B be nonempty closed disjoint subsets of X. Suppose that for every $b \in B$ there is a continuous function $f_b : X \to [0, 1]$ such that $f_b(b) = 1$ and $f_b(x) = 0$ for all $x \in A$. Prove that there exist open sets U and V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

Exam END

GRADUATE PROGRAM IN MATHEMATICS

RUTGERS UNIVERSITY

Written Qualifying Examination

Day 2, January, 2012

This examination will be given in two three-hour sessions, today being the second. At each session the examination has two parts.

- Answer all three of the questions in Part I (numbered 1–3), and answer three of the six questions in Part II (numbered 4–9).

Before handing in your exam,

- Be sure your ID is on each book you are submitting.
- Label the books at the top as Book 1 of X, Book 2 of X, etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

If you work on more than three questions in Part II, indicate clearly which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day–Part I: Answer each of the following three questions.

1. Suppose that A is a 5×5 real matrix and $(A - 2I)^5 = 0$.

(a) What Jordan canonical forms are possible for A?

(b) Suppose that B is another 5×5 real matrix, $AB = BA$, and the minimal polynomial of B is $t^3 + t$. Now which of your answers to (a) are still possible Jordan canonical forms for A? Explain your reasoning.

2. Let U be the strip given by $U = \{x + iy \in \mathbb{C} : -1 < x < 1\}$. Let

$$F(z) = \int_0^\infty \frac{u^z}{\sqrt{1+u^4}}\, du.$$

(Here by definition, $u^z = e^{z \ln u}$.) Prove that the integral converges absolutely for $z \in U$, and that F is holomorphic on U. Explain your reasoning.

3. (a) In the context of Lebesgue measure on $\mathbb{R}$, state Fatou's Lemma carefully and completely.

(b) For all $n \geq 1$, let $f_n : \mathbb{R} \to [0, \infty)$ be a Lebesgue-measurable function. Assume that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$, and $\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, \mathrm{d}\mu = \int_{\mathbb{R}} f \, \mathrm{d}\mu < \infty$. For every Lebesgue-measurable set E, prove that

$$\lim_{n\to\infty} \int_E f_n \, \mathrm{d}\mu \text{ exists and } \lim_{n\to\infty} \int_E f_n \, \mathrm{d}\mu = \int_E f \, \mathrm{d}\mu.$$

(c) Let all hypotheses of (b) hold, except assume that $\int_{\mathbb{R}} f \, \mathrm{d}\mu = \infty$. Give an example to show that the conclusion in part (b) may fail.

The exam continues on the next page

Second Day–Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let X be a set, $\mathcal{M}$ a σ-algebra of subsets of X, and $\mu : \mathcal{M} \to [0, \infty]$ a countably additive measure.

(a) Assume that $\theta_n : X \to \mathbb{C}$ and $\psi_n : X \to \mathbb{C}$ are μ-measurable for $n \geq 1$. Also assume that $\theta_n \to 0$ and $\psi_n \to 0$ in measure, where 0 is the zero function. Prove that $f_n \to 0$ in measure, where f_n is the function $f_n(x) = \theta_n(x)\psi_n(x)$. Include the definition of "$f_n \to 0$ in measure."

(b) Still assuming that $\theta_n \to 0$ in measure, let $g : X \to \mathbb{C}$ be any measurable function. If $\mu(X) < \infty$, prove that $h_n \to 0$ in measure, where h_n is the function $h_n(x) = \theta_n(x)g(x)$. Show by example that the hypothesis $\mu(X) < \infty$ is necessary.

5. Let f be an entire function which is one-to-one. Prove that there exist complex constants a, b such that $a \neq 0$ and $f(z) = az + b$ for all $z \in \mathbb{C}$. (Recall that a function f, defined on a set A, is one-to-one iff, whenever $a, b \in A$ and $f(a) = f(b)$, it follows that $a = b$.)

6. Let k be any field and let $R = k[t]$. Prove that there are rings R_1 and R_2, not isomorphic to each other, such that for any quadratic polynomial $q \in R$, the quotient ring $R/(q)$ either is a field or is isomorphic to R_1 or R_2.

7. Give an example of a nonzero holomorphic function $f : D \to \mathbb{C}$ (where D is the unit disc $\{z : |z| < 1\}$) such that f has infinitely many zeros in D.

The exam continues on the next page

8. Suppose that H is a subgroup of the group G and n is an integer such that $|G : H| < n$. Let x be an element of G of order n. Show that if n is prime, then necessarily $x \in H$. Give an example in which n is not prime and the conclusion fails.

9. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose that $\{f_n\}$ is a sequence of real valued measurable functions on X, and that the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for almost every x. Suppose that $\int_X |f(x)|\, \mathrm{d}\mu > 0$.

(a) Show that for some $\epsilon > 0$, there is a strictly positive number b so that for all *sufficiently large* n there is a set E_n with $\mu(E_n) > \epsilon$ and $|f_n(x)| > b$ for all $x \in E_n$.

(b) Is it always possible to choose the set E_n to be independent of n for sufficiently large n? Explain why or why not.

Exam END