RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2013, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts.

Before starting your exam,

- You will be given an anonimizer (a secret ID) to be used on your answer books (so that your real identify is not revealed to the grading committee). Be sure your secret ID, not your real name, is on each book you are submitting. The same secret ID will be used for both days, so please keep it safe.
- Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9).
- If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam,

- Label the books at the top as Book 1 of X, Book 2 of X, etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

First Day—Part I: Answer each of the following three questions

1. Let p be a prime number. Prove that the ring $\mathbb{Z}/p^2\mathbb{Z}$ is not isomorphic to a product of two nontrivial rings.

2. For the series

$$f(z) = \sum_{n=1}^{\infty} \sqrt{n} z^n \,, \tag{1}$$

prove that

(i) (1) defines a holomorphic function f in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$,

(ii) this holomorphic function cannot be extended to a continuous complex-valued function on the closed unit disc $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$.

3. Consider the system of equations

$$x^2u + y^2v - xv^2 = 1$$
$$x + (yv)/u + v^2 = 3$$

for the real variables x, y, u, v.
(i) State precisely a version of the Implicit Function Theorem, and use it to prove that there exist a sufficiently small neighborhood Z of $(x, y) = (1, 1)$ in $\mathbb{R}^2$ and continuously differentiable functions $u = u(x, y)$ and $v = v(x, y)$ defined in Z with $u(1, 1) = v(1, 1) = 1$, and a neighborhood W of $(u, v) = (1, 1)$ such that, for any $(x, y) \in Z$, the only solution (x, y, u, v) to the system in $Z \times W$ is given by $(x, y, u(x, y), v(x, y))$.
(ii) Find the value of $\partial u/\partial x$ at the point $x = 1, y = 1$.

The exam continues on next page

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let $f(x,y)$ be a real-valued function defined on the unit square $0 \le x, y \le 1$.

(i) Assume that f is continuous in x for each fixed y and continuous in y for each fixed x. Show that f is Lebesgue measurable.

(ii) Assume that f is continuous in x for each fixed y. Does this imply that f is Lebesgue measurable? Justify your answer.

5. Suppose that A is a square complex matrix and f is a polynomial in $\mathbb{C}[t]$ such that $f(A)$ is diagonalizable. If $f'(A)$ is invertible, where f' is the derivative of f, prove that A is diagonalizable in $\mathbb{C}$.

6. Prove that if u is a bounded harmonic function on the half-plane $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$ such that $\lim_{x \to x_0, y \to 0} u(x,y) = 0$ for every $x_0 \in \mathbb{R}$, then $u \equiv 0$.

7. Prove that if f is an entire function such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x+iy)|^2 dx\, dy < \infty$$

then $f \equiv 0$.

8. For real-valued measurable functions f and g on $(0, \infty)$, let

$$F(x) = \int_0^\infty f\big(\frac{x}{y}\big) g(y) \frac{dy}{y}.$$

If $1 \le p < \infty$, set

$$[h]_p = \left(\int_0^\infty |h(x)|^p \frac{dx}{x} \right)^{1/p},$$

and for $p = \infty$, define $[h]_\infty = \operatorname{ess\,sup} |h|$. Prove that for $1 \le p \le \infty$,

$$[F]_p \le [f]_1\, [g]_p.$$

9. We say that a group action of a group G on a set Ω is without fixed points if for each $\alpha \in \Omega$ there exists $g \in G$ such that $g\alpha \ne \alpha$. Let p and q be distinct primes and let G be any group of order pq. Show that for every integer $n \ge pq$ there exists a set Ω of cardinality n and a group action of G on Ω without fixed points.

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2013, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts.

Before starting your exam,

- You have been given an anonimizer (a secret ID) on Day 1 to be used on your answer books (so that your real identify is not revealed to the grading committee). Be sure your secret ID, not your real name, is on each book you are submitting.
- Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9).
- If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam,

- Label the books at the top as Book 1 of X, Book 2 of X, etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

Second Day—Part I: Answer each of the following three questions

1. Let $f \in L^1(-\infty, \infty)$, and let $h > 0$ be fixed. Prove that

$$\frac{1}{2h}\int_{-\infty}^{\infty}\int_{x-h}^{x+h} f(y)dy\,dx = \int_{-\infty}^{\infty} f(x)dx.$$

2. Let L be a linear operator in an n-dimensional linear space E and suppose that there is a basis in E such that L is represented by a matrix $A = (a_{ij})$ where $a_{ij} = \delta_{i+1,j}$ for $i, j = 1, 2, \ldots, n$. Is there a basis in E such that L is represented by a diagonal matrix?

3. Prove that if U is an open connected subset of $\mathbb{C}$ and $\mathbf{f} = \{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on U such that $\mathrm{Re} f_n(z) > 0$ for every n and every $z \in U$, then $\mathbf{f}$ has a subsequence $\{f_{n(k)}\}_{k=1}^{\infty}$ such that either

(i) $f_{n(k)}(z)$ converges uniformly on compact subsets of U to a holomorphic function $f(z)$,

or

(ii) $f_{n(k)}(z)$ converges to ∞ (that is, $|f_{n(k)}(z)| \to +\infty$) uniformly on compact subsets of U.

Make sure that your proof indicates clearly where and how you are using the assumption that U is connected.

The exam continues on next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let V be a finite-dimensional vector space over a field F, and let $T : V \to V$ be a linear transformation. Suppose that for every $v \in V$ there exists an irreducible polynomial $f_v \in F[t]$ and a positive integer n_v such that $f_v(T)^{n_v}(v) = 0$. Prove that there is an irreducible polynomial $f \in F[t]$ and a positive integer n such that for all $v \in V$, $f(T)^n(v) = 0$.

5. Evaluate the integral

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx\,,$$

where α is a real number such that $0 < \alpha < 1$.

6. Let $E \subset (-\infty, \infty)$ be a Lebesgue measurable set which has the point $x = 0$ as a point of density. For $r > 0$, define

$$E_r = E \cap (-r, r) \quad \text{and} \quad S_r = \{2x - y : x, y \in E_r\}.$$

Show that $(-r, r) \subset S_r$ if r is sufficiently small. (Reminder: 0 is a point of density of E if $|E_r|/(2r) \to 1$ as $r \to 0$.)

7. Let A and B be finite groups. Prove that any Sylow p-subgroup of the direct product $A \times B$ is a product of a p-subgroup of A and a p-subgroup of B.

8. Let $k(y) = c \exp\{-y^2\}$, $-\infty < y < \infty$, where c is chosen so that $\int_{-\infty}^{\infty} k(y)dy = 1$. If $f \in L^1(-\infty, \infty)$, prove that, for every point x of the Lebesgue set of f, there holds

$$\lim_{t\to 0+} \int_{-\infty}^{\infty} f(x-y)\,\frac{1}{t}k(\frac{y}{t})\,dy = f(x).$$

9. Construct a biholomorphic map (i.e., a holomorphic map with a holomorphic inverse) from the sector

$$S = \{re^{i\theta} : 0 < r < 1 \quad \text{and} \quad 0 < \theta < \frac{\pi}{3}\}$$

onto the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Exam Day 2 End