RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Jan. 2016

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in the order that they appear in the book.

Part I. Answer all questions.

1. Let $P(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients, and assume that $P(0)$ and $P(1)$ are odd integers. Prove that $P(x)$ has no integer roots.

2. Let M denote the additive group $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ and let $\mathrm{End}(M)$ denote the set of homomorphisms $\phi : M \to M$. Show that $\mathrm{End}(M)$ is infinite and noncommutative.

3. Let $A \in M_n(\mathbb{C})$ be a matrix such that $A^k = A$ for some integer $k \geq 2$. Prove that A is diagonalizable.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let F be a field whose multiplicative group F^* is cyclic. Prove that F is finite.

5. Let G be a group of order 108. Prove that G is not simple.

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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Part I. Answer all questions.

1. Let $\xi > 0$ and compute

$$\int_{\mathbb{R}} \frac{\cos(2\pi x\xi)}{4x^2+1}\,dx$$

using a contour shift.

2. Let $\mathbb{D}$ denote the open unit disc $\{z : |z| < 1\} \subset \mathbb{C}$, and suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \;\leq\; |f(z)| \;\leq\; \frac{|f(0)| + |z|}{1 - |f(0)||z|}.$$

3. Consider the vector field $F(x,y,z) = (-4xz^3, 0, z^4)$ in $\mathbb{R}^3$ and let S be the (compact) portion of the paraboloid $z = x^2 + y^2$ having $z \leq 9$. Use Stokes' theorem to evaluate

$$\iint_S F(x,y,z) \cdot d\mathbf{S}\,,$$

where $d\mathbf{S}$ is the vector surface element corresponding to the upward pointing normal vector.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Suppose f is an entire function with $\iint_{\mathbb{C}} |f(z)|^2 dxdy < \infty$. Show that $f(z) = 0$ for all $z \in \mathbb{C}$.

5. Find the Riemann mapping f from $\{z \in \mathbb{C} \,|\, |z| < 1\}$ onto $\mathbb{C} - \{x \in \mathbb{R} \,|\, x \leq -\frac{1}{4}\}$ with the properties that $f(0) = 0$ and $f'(0) = 1$.

End of Session 2

RUTGERS UNIVERSITY

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Written Qualifying Examination

Jan. 2016

Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in the order that they appear in the book.

Part I. Answer all questions.

1. A subset A of $\mathbb{R}^n$ is said to be *path-connected* if, given any two points $x_0, y_0 \in A$, there exists a continuous path $\phi : [0,1] \longrightarrow A$ such that $\phi(0) = x_0$ and $\phi(1) = y_0$.

a) Prove that if $A \subset \mathbb{R}^n$ is non-empty and path-connected, then A is connected.

b) Suppose now that A is an open subset of $\mathbb{R}^n$. For $x \in A$, let C_x be the set of points z in A for which there is a continuous path in A from x to z. Prove that C_x is open in A. (Hint: use the fact that every ball in $\mathbb{R}^n$ is path-connected, and use composition of paths.)

c) Continuing with the assumptions of part b), prove that for any two points $x, y \in A$, either $C_x = C_y$ or $C_x \cap C_y = \emptyset$.

d) Continuing with the assumptions of parts b) and c), prove that if A is connected, then A is also path-connected. (Hint: use the fact that A can be written as $\cup_{x \in A} C_x$.)

2. Let $[a,b]$ be a (bounded) interval of $\mathbb{R}$ and let m be Lebesgue measure. Let M be a positive real number and let $f_1, f_2, \ldots$ be a sequence of measurable functions on $[a,b]$ for which $\int_a^b |f_n| dm \le M$ for every n. Assume that $f_n(x) \to f(x)$ as $n \to \infty$ for m-almost every x.

a) State Fatou's lemma.

b) Show that $\int_a^b |f| \, dm \le M$.

c) Suppose that $\|f_n - f\|_1 \to 0$. Prove for every $\epsilon > 0$ there exists $\delta > 0$ such that if $A \subset [a,b]$ is m-measurable and $m(A) \le \delta$, then $\int_A |f_n| dm \le \epsilon$ for all n.

3. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ satisfy:

(i) for each $x \in [0,1]$, the function $y \mapsto f(x,y)$ is Riemann integrable on $[0,1]$; and

(ii) for each $y \in [0,1]$, the function $x \mapsto f(x,y)$ is Borel measurable.

Show that the function $g(x) := \int_0^1 f(x,y) \, dy$ is Borel measurable.

(Note: In general, f will not be Borel measurable as a function on $[0,1] \times [0,1]$.)

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. a) Define Lebesgue outer measure $m^*(A)$ for subsets A of $\mathbb{R}$.

b) If $B \subset \mathbb{R}$ and α is a positive real number, define $\alpha B = \{\alpha x |\, x \in B\}$. Show that $m^*(\alpha B) = \alpha\, m^*(B)$.

c) Show from the definition of Lebesgue integral that for any positive, Lebesgue measurable function f and positive real number α,

$$\int f(x/\alpha)\, m(dx) = \alpha \int f(x)\, m(dx).$$

5. Let m denote Lebesgue measure on $\mathbb{R}$ and let m^2 denote Lebesgue measure on $\mathbb{R}^2$. Let $f \in L^1(\mathbb{R})$.

a) Show that $h(x, y) = f(x)f(x + y) \in L^1(\mathbb{R}^2, m^2)$.

b) Show that for Lebesgue almost every y, $x \mapsto f(x)f(x + y)$ defines a function in $L^1(\mathbb{R})$.

c) Give an example of a function $g \in L^1(\mathbb{R}, m)$ for which $x \to g(x)g(x+y)$ is **not** in $L^1(\mathbb{R})$ for at least one $y \in \mathbb{R}$.