

RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination
August 2016

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in the order that they appear in the book.

Part I. Answer all questions.

1. Let G be an abelian group and for each positive integer n , define

$$G[n] = \{g \in G \mid ng = 0\}.$$

- (a) Show that if m and n are positive integers and m divides n , then $G[m] \subseteq G[n]$, and $G[n]/G[m]$ is isomorphic to a subgroup of $G[n/m]$.
(b) Give an example in which m divides n but $G[n]/G[m] \not\cong G[n/m]$.
Prove your assertion.

2. Let T be a square matrix over \mathbb{C} .

- (a) Show that if T is invertible and T^k is diagonalizable for some positive integer k , then T is diagonalizable.
(b) Show that the invertibility hypothesis cannot be omitted in (a).

3. Let I be an ideal in a principal ideal domain R . Show that if $I \neq R$, then

$$\bigcap_{n=1}^{\infty} I^n = (0).$$

(Here I^n is the ideal generated by all products $x_1 \cdots x_n$ such that $x_i \in I$ for all $i = 1, \dots, n$.)

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let B be a nondegenerate symmetric bilinear form on a 2-dimensional vector space V over the finite field F_p of p elements, where p is prime. Assume that $p \neq 2$. Show that there is always a vector $v \in V$ such that $B(v, v) = 1$.
5. Let G be a finite group acting transitively on a set Ω and suppose that $|\Omega| = p^m$ for some prime p and positive integer m . Let P be a Sylow p -subgroup of G (for the same prime p). Prove: P acts transitively on Ω .

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in the order that they appear in the book.

Part I. Answer all questions.

1. Use a contour integral to evaluate

$$\int_0^{2\pi} \frac{d\theta}{(2 + \cos(\theta))^2}.$$

2. Write $z = x + iy$ and let $R = \{(x, y) : (x - 1)^2 + y^2 < 1, y > x\}$. Find a biholomorphic map from R to the unit disk $D = \{(x, y) : x^2 + y^2 < 1\}$. You may express your answer as a composition of explicitly given biholomorphic maps.
3. Let $u(x, y)$ be a harmonic function on the unit disk $D = \{z : |z| < 1\}$; specifically, we assume that u is twice continuously differentiable on D and that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where $z = x + iy$.

(a) Show that $f(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$ is a holomorphic function on D .

(b) For any piecewise smooth curve $\gamma \subset D$ connecting 0 to $z \in D$, define $F(z) = \int_\gamma f(z) dz$. Prove that F is a well defined holomorphic function on D .

(c) Show that $\operatorname{Re}(F(z)) = u(z) - u(0)$.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let $f(z)$ be a holomorphic function on the punctured disk

$$D_0 = \{z : 0 < |z| < 1\}.$$

Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent expansion of $f(z)$.

- (a) Prove that for any $0 < r < 1$,

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 r^{2n}.$$

[This is an instance of Parseval's theorem, which you may not quote.]

- (b) Prove that if $\int_{D_0} |f(z)|^2 dA < \infty$, then $f(z)$ has a removable singular point at 0. Here dA is the Euclidean area element in \mathbb{R}^2 .

5. Suppose f and g are holomorphic in a region containing the closed unit disc $\overline{D} = \{z : |z| \leq 1\}$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in \overline{D} . Let

$$f_t(z) = f(z) + tg(z).$$

Show that if $t > 0$ is sufficiently small, then

- (a) $f_t(z)$ has a unique zero in \overline{D} , and
(b) if z_t is this zero, then the mapping $t \rightarrow z_t$ is continuous.

End of Session 2

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Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

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Part I. Answer all questions.

1. Let X denote the set of all continuous real-valued functions $f : [0, 1] \rightarrow \mathbb{R}$. For $f, g \in X$, define

$$d(f, g) = \max \left\{ |f(x) - g(x)| : 0 \leq x \leq 1 \right\}.$$

- a. **Prove** that d is a metric on X and that (X, d) is a complete metric space.
- b. Let $\mathbf{0}$ denote the function in X which is identically equal to zero, and let $B = \{f \in X : d(f, \mathbf{0}) \leq 1\}$. **Prove** that B is not compact. HINT: In a metric space, compactness is equivalent to sequential compactness.
2. Let a, b be real numbers such that $a < b$, and let $f : [a, b] \rightarrow \mathbb{R}$.
- a. **Define** what it means for f to be “absolutely continuous on $[a, b]$ ” (this is an ε - δ definition).
- b. State a theorem relating the absolute continuity of such a function to its differentiability.
- c. Assume that the restriction $f|_{[a, a+\varepsilon]}$ is absolutely continuous for every ε such that $0 < \varepsilon < 1$, and that $\int_0^1 x^2 |f'(x)|^p dx < \infty$ for some real number p such that $p > 3$. Prove that $\lim_{x \rightarrow 0} f(x)$ exists and is finite. (HINT: Prove that $\int_0^1 |f'(x)| dx < \infty$.)
3. Let n be a positive integer.

- a. **Define** what it means for a subset S of \mathbb{R}^n to be “connected”.
- b. Let Ω be an open connected subset of \mathbb{R}^n . Let $f : \Omega \rightarrow \mathbb{R}$ be a function such that

$$\lim_{\varepsilon \rightarrow 0} \frac{f(p + \varepsilon v) - f(p)}{\varepsilon} = 0$$

for every $p \in \Omega$ and every $v \in \mathbb{R}^n$. **Prove** that f is a constant. Make sure that you use in this proof the definition of “connected” that you gave in Part a.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function such that

$$\int_{-\infty}^{\infty} (1 + |x|)|f(x)|dx < \infty.$$

Define

$$g(y) = \int_{-\infty}^{\infty} f(x) \cos(xy)dx.$$

1. **Prove** that g is continuously differentiable (that is, prove that the derivative $g'(y) = \lim_{h \rightarrow 0} \frac{g(y+h) - g(y)}{h}$ exists for every $y \in \mathbb{R}$, and is a continuous function of y).
 2. **Write a formula** for g' , as an integral.
5. Let T be a real number such that $T > 0$. Let $f : (0, T) \rightarrow \mathbb{R}$ be a Lebesgue integrable function. (Here $(0, T)$ is the open interval $\{x \in \mathbb{R} : 0 < x < T\}$.) Define a function $g : (0, T) \rightarrow \mathbb{R}$ by letting

$$g(x) = \int_x^T \frac{f(t)}{t} dt.$$

Prove that g is integrable on $(0, T)$ and $\int_0^T g(x)dx = \int_0^T f(x)dx$.

HINTS: (a) You may want to consider first the case in which f is nonnegative.

(b) Use the Fubini-Tonelli theorem.)

End of Session 3