Note: This problem set concentrates on material from the end of the course. **For a complete review, you should also study the review problem sets for the two in-class exams. Please consider these earlier problem sets as implicitly included with this one.** Particular topics that should be reviewed from earlier sets include: (i) Solving systems of linear equations, row operations, elementary matrices; (ii) The LU decomposition of a matrix; (iii) Inverses of matrices; (iv) Subspaces, finding bases for $\operatorname{Col} A$, $\operatorname{Row} A$, and $\operatorname{Null} A$; (v) Determinants and characteristic polynomial of a matrix.

1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$.

(a) State the Cauchy–Schwarz inequality and the triangle inequality for $\mathbf{u}$ and $\mathbf{v}$.

(b) Prove the triangle inequality from the Cauchy–Schwarz inequality by calculating $\|\mathbf{u}+\mathbf{v}\|^2$.

2. Suppose that $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$, and that $\mathbf{w}$ is a vector in $\mathbb{R}^3$ with $\|\mathbf{w}\| = 5$ and $\mathbf{w}\cdot\mathbf{u} = 13$.

(a) Compute $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\mathbf{u}\cdot\mathbf{v}$, and $\|\mathbf{u}+\mathbf{v}\|$.

(b) Show that the Cauchy-Schwarz and triangle inequalities are satisfied by $\mathbf{u}$ and $\mathbf{v}$.

(c) Compute $(\mathbf{u}+2\mathbf{w})\cdot(\mathbf{u}-\mathbf{w})$.

3. Let V be the subspace of $\mathbb{R}^3$ spanned by the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(a) Find the vector $\mathbf{y}$ that is the orthogonal projection of $\mathbf{x}$ onto V. Then calculate $\mathbf{z} = \mathbf{x} - \mathbf{y}$ and check that $\mathbf{z} \perp V$.

(b) Find a basis for $V^\perp$ (the subspace of vectors orthogonal to V). (*Hint:* This is the null space of a 1×3 matrix.)

(c) Use part (b) and Gram-Schmidt to obtain an orthonormal basis $\{\mathbf{q}_1,\ \mathbf{q}_2\}$ for $V^\perp$.

(d) Let $\mathbf{z}$ be the vector from (a). Then $\mathbf{z} \in V^\perp$, so $\mathbf{z} = c_1\mathbf{q}_1 + c_2\mathbf{q}_2$ for suitable coefficients c_1, $\mathbf{c}_2$. Give the general formula for these coefficients in terms of inner products, and use the formula to calculate the coefficients for this particular $\mathbf{z}$. Then check that $\mathbf{z} = c_1\mathbf{q}_1 + c_2\mathbf{q}_2$.

4. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

(a) Give the dimensions of $\operatorname{Row} A$, $\operatorname{Col} A$, and $\operatorname{Null} A$.

(b) Find orthonormal bases for $\operatorname{Row} A$, $\operatorname{Col} A$, and $\operatorname{Null} A$. *Hint:* One of these requires no calculation, one requires a small calculation, and one requires Gram-Schmidt.

5. Find a 3×3 orthogonal matrix Q with first column $\dfrac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Hint: There are some easy choices for columns 2 and 3.

6. True or false (four separate cases–justify your answer in each case). If a 4×4 matrix A satisfies the following condition, it is diagonalizable:

T F (a) the eigenvalues of A are 0, 1, 2, 3.

T F (b) the characteristic polynomial of A is $\lambda^2(\lambda-1)(\lambda-2)$;

T F (c) the eigenvalues of A are 0, 1, and 2, and A has rank 2;

T F (d) the eigenvalues of A are 0 and 2, and A is symmetric;

7. (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}$.

(b) Find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

8. A certain 3×3 matrix A has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$, and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(a) Use the formula $A = PDP^{-1}$ (for suitable P and D) to find A.

(b) Let $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Use (a) to find coefficients c_1, c_2, c_3 so that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Then compute $A^n\mathbf{x}$ from this formula for $\mathbf{x}$ for arbitrary $n > 0$. What is a good approximation to $A^n\mathbf{x}$ for n large?

9. Suppose that A is a symmetric $n \times n$ matrix and that the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ satisfy $A\mathbf{x} = 2\mathbf{x}$ and $A\mathbf{y} = 3\mathbf{y}$. Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

10. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & 4 \end{bmatrix}$.

(a) Find a vector $\mathbf{v} \in \mathbb{R}^3$ such that $A = \mathbf{v}\mathbf{v}^T$. Show that $\mathbf{v}$ is an eigenvector for A and find the eigenvalue.

(b) Calculate the nullity of A and find a basis for the zero eigenspace of A. Check that $\mathbf{v} \perp \mathrm{Null}(A)$ and explain why you know this without explicit calculation.

(c) Use (a) and (b) to find an orthonormal set of eigenvectors of A which form a basis for $\mathbb{R}^3$.

(d) Find an orthogonal matrix Q and a diagonal matrix D such that such that $A = QDQ^T$.

11. Classify each statement as true (T) or false (F). If your answer if T, **give a brief proof** showing that the statement is *always* true; if your answer is F, **give a specific example** for which the statement is not true.

T F (a) The null space of a matrix A is the orthogonal complement of the column space of A.

T F (b) Every orthogonal matrix has null space $\{\mathbf{0}\}$.

T F (c) If P and Q are orthogonal matrices then P^TQ is an orthogonal matrix.

T F (d) If A is an $n \times n$ matrix and 0 is an eigenvalue of A then $\mathrm{Col}\, A \neq \mathbb{R}^n$.

T F (e) If Q is an orthogonal matrix then $Q = Q^{-1}$.

T F (f) If A is an $n \times n$ matrix then eigenvectors for distinct eigenvalues of A are orthogonal.

12. Suppose that W is a subspace of $\mathbb{R}^n$ of dimension k and that $\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ such that $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis for W.

(a) Any vector $\mathbf{u} \in \mathbb{R}^n$ has an expansion $\mathbf{u} = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n$. Give a simple formula for the coefficients c_j in terms of inner products.

(b) We know that any $\mathbf{u} \in \mathbb{R}^n$ can be written uniquely as $\mathbf{u} = \mathbf{w} + \mathbf{z}$, with $\mathbf{w} \in W$ and $\mathbf{z} \in W^\perp$. Explain why $\mathbf{w} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k$.

(c) Let C be the $n \times k$ matrix with columns $\mathbf{w}_1, \ldots, \mathbf{w}_k$. Then $W = \mathrm{Col}(C)$. Show that $C^TC = I_k$. Then *using your answers to (a) and (b)*, show that P_W, the orthogonal projection matrix onto W, is given by $P_W = CC^T$. (Recall that, in the notation of (b), $\mathbf{w} = P_W\mathbf{u}$.)

(d) Derive the result in (d) from the general formula for P_W in terms of C.

13. Consider the data points $(-3, 9)$, $(-1, 7)$, $(0, 5)$, $(4, 1)$ in the (x, y) plane..

(a) The method of least squares for a straight line fit to this data minimizes a certain quantity. What is that quantity in this case? Give the answer explicitly; define any variables used.

(b) We obtain a solution by solving the *normal equations* $C^TC\mathbf{u} = C^T\mathbf{y}$. What is C for the data above? What is $\mathbf{y}$? What is $\mathbf{u}$?

(c) Find the equation of the straight line which best fits this data.

14. Do the True-False questions from Sections 6.1 through 6.6 that are listed in the homework assignments.