1. See the textbook.

2. See the textbook.

3. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 8 \end{bmatrix}$. Let $\mathbf{b} = \begin{bmatrix} 3 \\ -4 \\ 10 \\ 12 \end{bmatrix}$. Solve triangular system $L\mathbf{y} = \mathbf{b}$ starting with y_1: $\mathbf{y} = \begin{bmatrix} 3 \\ -4 \\ -4 \\ 16 \end{bmatrix}$. Then solve $U\mathbf{x} = \mathbf{y}$ starting with x_4: $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$.

4. (a) $\operatorname{rank} A = \dim \operatorname{Col} A = 3$ (# pivot columns), $\operatorname{nullity} A = \dim \operatorname{Null} A = 2$ (# columns $-$# pivot columns), $\operatorname{rank} A^T = \dim \operatorname{Row} A = 3$ (# nonzero rows in R) $\operatorname{nullity} A^T = \dim \operatorname{Null} A^T = 4 - 3 = 1$
(b) basis for $\operatorname{Col} A$: columns #1, #3, and #4 of A basis for $\operatorname{Row} A$: nonzero rows of R

basis for $\operatorname{Null} A$: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ -4 \\ 1 \end{bmatrix}$ (take free variables x_2, x_5 either 1 or 0)

5. (a) True: Let A be $n \times n$. Since $\operatorname{nullity} A = 0$, the columns of A are linearly independent. Since there are n columns, they give a basis for $\mathbb{R}^n$. Hence A is invertible and the solution is $\mathbf{x} = A^{-1}\mathbf{b}$.
(b) False: The correct condition for invertibility is $\det A \neq 0$.
(c) False: Since $\mathbf{b} \neq 0$, the zero vector is not in the set.
(d) True: $\dim \operatorname{Row} A \leq m < n$, so $\operatorname{nullity}(A) = n - \operatorname{rank} A \geq 1$.
(e) True: $\dim \operatorname{Row} A = \operatorname{rank} A$, and $\operatorname{rank} A + \operatorname{nullity} A = n$
(f) False: $\operatorname{nullity} A^T = \#$ rows of $A - \operatorname{rank} A$.
(g) True: A is not invertible since $\operatorname{rank} A < n$. Hence $\det A = 0$.
(h) False: $1 \leq \dim W \leq r$, since there must be a nonzero eigenvector in W.
(i) True: Take an eigenvector for each eigenvalue. This is a linearly independent set of vectors (Theorem 5.3). This set has n vectors, hence it is a basis for $\mathbb{R}^n$. Thus A is diagonalizable (Theorem 5.2).

6. (a) Span of a set of vectors is always a subspace. The given vectors are independent, so they are a basis for W and $\dim W = 2$.
(b) $W = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -8 \\ 1 \end{bmatrix} \right\}$, so W is a subspace and $\dim W = 2$ as in (a).
(c) $W = \left\{ t \begin{bmatrix} 1 \\ -8 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$, so W is a subspace and $\dim W = 1$ as in (a). Note that the two parameters r, s only appear in the combination $t = r + s$.
(d) W is not a subspace since it does not contain the zero vector.
(e) $W = \mathbb{R}^3$ and $\dim W = 3$ (use any basis).
(f) W is the zero subspace and $\dim W = 0$. It has no basis (only nonzero subspaces can have a basis, since the vectors in a basis must be linearly independent).

7. (a) By cofactors of row 1:

$$1 \det \begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & 2 \\ -3 & -2 & 1 \end{bmatrix} - (-1) \det \begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & -2 & 1 \end{bmatrix} + 1 \det \begin{bmatrix} 2 & 1 & 4 \\ 1 & -2 & 2 \\ 3 & -3 & 1 \end{bmatrix} - 0 \det \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & -3 & -2 \end{bmatrix}$$

$$= 1 \cdot (21) - 1 \cdot (-5) + 1 \cdot (25) + 0 \cdot (-28) = 51$$

(b) By cofactors of row 2:

$$-2 \det \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 2 \\ -3 & -2 & 1 \end{bmatrix} + 1 \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 3 & -2 & 1 \end{bmatrix} - 3 \det \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 2 \\ 3 & -3 & 1 \end{bmatrix} + 4 \det \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & -3 & -2 \end{bmatrix}$$

$$= 2 \cdot (9) + 1 \cdot (10) + 3 \cdot (1) + 4 \cdot (5) = 51$$

(c) Use elementary row operations (in this order): $\mathbf{r}_2 - 2\mathbf{r}_1 \to \mathbf{r}_2$, $\mathbf{r}_3 - \mathbf{r}_1 \to \mathbf{r}_3$, $\mathbf{r}_4 - 3\mathbf{r}_1 \to \mathbf{r}_4$, $\mathbf{r}_3 + (1/3)\mathbf{r}_2 \to \mathbf{r}_3$, $\mathbf{r}_4 + 15\mathbf{r}_3 \to \mathbf{r}_4$ to transform A into $U = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 51 \end{bmatrix}$. Then $\det A = \det U = 1 \cdot 3 \cdot (1/3) \cdot 51 = 51$.

8. (a) Use row operations $A = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix} \to \begin{bmatrix} \mathbf{c} \\ \mathbf{b} \\ \mathbf{a} \end{bmatrix} \to \begin{bmatrix} \mathbf{c}+3\mathbf{b} \\ \mathbf{b} \\ \mathbf{a} \end{bmatrix} \to \begin{bmatrix} \mathbf{c}+3\mathbf{b} \\ 2\mathbf{b} \\ \mathbf{a} \end{bmatrix} = B$. The changes in determinant are $\det A = 5 \to -5 \to -5 \to 2(-5) = \det B$.
(b) $\det C = (1)(2)(-2) = -4$ since C is triangular and $\det A^T = \det A = 5$. Hence
$\det AC^3A^T = (\det A)^2(\det C)^3 = 5^2 \cdot (-4)^3 = -1600$.

9. (a) Characteristic polynomial $\det \begin{bmatrix} 2-t & -1 \\ 4 & -3-t \end{bmatrix} = t^2 + t - 2 = (t-1)(t+2)$.
Eigenvalues/Eigenvectors:
$\lambda_1 = 1$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = -2$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ (or any nonzero multiples of these vectors).
(b) Take $P = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

10. (a) $A\mathbf{v} = (\mathbf{v}\mathbf{v}^T)\mathbf{v} = \mathbf{v}(\mathbf{v}^T\mathbf{v}) = 14\mathbf{v}$ since $\mathbf{v}^T\mathbf{v} = [\,1 \quad 2 \quad 3\,] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 14$. The eigenvalue is 14.

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} = [\,\mathbf{v} \quad 2\mathbf{v} \quad 3\mathbf{v}\,]$. (c) Since $\operatorname{rank} A = \dim \operatorname{Col} A = 1$, we have $\dim \operatorname{Null} A = 2$.

(d) characteristic polynomial $\det \begin{bmatrix} (1-t) & 2 & 3 \\ 2 & (4-t) & 6 \\ 3 & 6 & (9-t) \end{bmatrix} = -t^2(t-14)$

Eigenvalues are $\lambda_1 = 14$ (multiplicity 1) and $\lambda_2 = 0$ (multiplicity 2).
(e) Let $\mathbf{u}_1$, $\mathbf{u}_2$ be a basis for Null A. They give a basis for the 0 eigenspace of A. Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}\}$ is an eigenvector basis for $\mathbb{R}^3$. Hence A is diagonalizable.
(f) Take $P = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{v}\,]$. Then the diagonal entries of D are $0, 0, 14$.

11. (a) The characteristic polynomial is $(1-t)^2(4-t)$ and the eigenvalues are 1 (algebraic multiplicity 2) and 4 (algebraic multiplicity 1)
(b) For eigenvalue $\lambda_1 = 1$:

$$A - \lambda_1 I_3 = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.\ \text{dimension eigenspace} = 1, \text{ basis } \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

For eigenvalue $\lambda_2 = 4$:

$$A - \lambda_2 I_3 = \begin{bmatrix} -3 & 2 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\ \text{dimension eigenspace} = 1, \text{ basis } \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(c) A is not diagonalizable, since there are not enough eigenvectors to give a basis for $\mathbb{R}^3$.