

Calculus at Rutgers

This edition of Jon Rogawski's text, *Calculus Early Transcendentals*, is intended for students to use in the three-semester calculus sequence Math 151/152/251 beginning with Math 151 in the fall 2007 semester.

The next few pages provide useful local information for the students in these courses. All of this information is also available using links from the **Math 151 web page**. Please see the link here:

<http://www.math.rutgers.edu/courses/>

Course descriptions

The courses which this text supports have these descriptions:

01:640:151-152 Calculus for Mathematical and Physical Sciences (4,4)

Math 151: Analytic geometry, differential calculus with applications, logarithmic and exponential functions, introduction to the integral, with additional theory and numerical applications. Math 152: Techniques of integration, elementary differential equations, sequences, infinite series, Taylor series, parametric equations, polar coordinates.

For mathematics, physics, computer science, statistics, chemistry, or engineering majors. Prerequisite for 151: 01:640:112 or 115 or appropriate performance on the placement test in mathematics.

01:640:251 Multivariable Calculus (4)

Analytic geometry of three dimensions, partial derivatives, optimization techniques, multiple integrals, vectors in Euclidean space, and vector analysis.

Honors sections of each of these courses (151H/152H/251H) and a parallel sequence of honors courses (191/192/291) are also offered. Permission of the Department is needed for admission to these sections and courses.

The three-semester calculus track

The three course sequence 151/152/251 is suitable for students who major in subjects which require more rigorous knowledge of calculus than can be obtained through other courses such as 640:135. For example, Math 251 is needed as a prerequisite for the physical chemistry courses at Rutgers. Physical chemistry is needed for certain biological science majors. Also students in economics and finance who intend more advanced study in quantitative areas would be better served with the 151/152/251 sequence than by limiting their study of calculus to Math 135 (Calculus for liberal arts majors).

This sequence of courses is prerequisite to most upper-level courses in all engineering majors and in physics, statistics, and mathematics. Certain courses in some majors (for example, marine sciences and meteorology) also directly require all of the 151/152/251 sequence, in addition to those with indirect requirements, such as courses or majors which need physical chemistry.

Technology

The presence and growing strengths of computational devices have changed mathematics and all of its applications in the last half-century. Hand-held devices such as graphing calculators are generally available and now almost cheap. Students in 151/152/251 should own graphing calculators.

Graphing calculator use

Students will be expected to know how to graph functions so that, in Math 151, the important results of the course can be reinforced visually. Several root-finding algorithms are studied in Math 151, and these can best be tested and understood with a calculator, even for simple functions. Criteria for learning when functions are increasing and decreasing are also discussed, and examples regarding these concepts can be checked with a graphing calculator.

In Math 152, many of the computations are elaborate, and, in practice, almost everyone uses calculators and computers to help. Students will implement approximate integration schemes and will investigate sequences and series with their calculators. Some of the work and answers to homework problems, even if symbolic, can frequently be checked using graphing calculators. Certainly many of the workshop problems can best be investigated using graphing calculators and considering examples.

Caution!

Graphing calculators are relatively simple devices and can be “fooled” fairly easily. Professor Mark McClure’s web page, which is currently at

<http://facstaff.unca.edu/mcmclur/graphing/graphs.html>

shows graphs generated by commonly used graphing calculators. The graphs have disturbing errors, some obvious and some subtle. All students should glance at the pictures displayed there.

Professor David Rusin’s web page, currently at

http://www.math.niu.edu/~rusin/known-math/99/calc_errors

contains useful discussion of the more serious difficulties which can occur with systematic use of calculators, and even with the more powerful symbolic algebra systems including Maple which is used in Math 251.

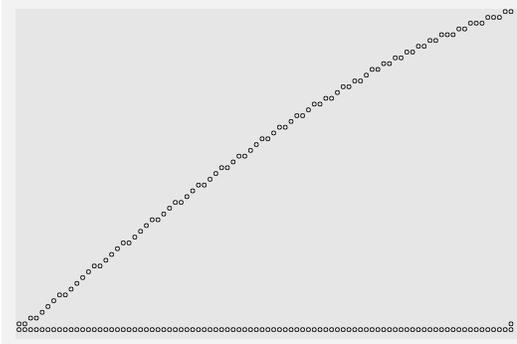
An example of a calculus idea and calculator support

Here’s a concept from calculus:

A function $f(x)$ defined on the interval $[a, b]$ is *strictly increasing* if for any x_1 and x_2 in $[a, b]$ with $x_1 < x_2$, then $f(x_1) < f(x_2)$.

The specific function $f(x) = x^2 - 2^x$ is strictly increasing on the specific interval $[2, 3]$. Verification of this assertion using the definition stated would mean checking an infinite number of statements about pairs of numbers x_1 and x_2 and the relationships between the values of $f(x_1)$ and $f(x_2)$. We can investigate this function on the interval $[2, 3]$ using a graphing calculator.

To the right is a picture (what's called a "screen-shot") of the TI-83 graph of $y = x^2 - 2^x$ with x between 2 and 3 and y between 0 and 1. The graph is made of 87 dots (yes, eighty-seven). The calculator computed 87 values of the function, and assigned these values to 55 different heights. There's definitely only a finite amount of graphical information in this picture.



The picture is almost (!) *consistent* with the statement that $x^2 - 2^x$ is increasing in the interval $[2, 3]$. The picture (these 87 dots!) is actually not increasing – some of the dots are on the same horizontal line (of course: 87 x 's and only 55 y 's and $87 > 55!$). The original "increasing" statement concerns an infinite number of inequalities about numbers. Verification that the function is increasing is easy with results from Math 151, but the TI-83 picture alone is not sufficient to verify the assertion.

Graphing calculators are wonderful, quick tools to drive intuition, but are unlikely to help with detailed verification.

Maple in Math 251

Much more sophisticated technology is now available. Although hand-held devices are "converging" to what can now be done on PC's, the standard is still such programs as **Maple**, **Mathematica**, and **Matlab**. Rutgers has a general site license for **Maple** so it is available on many university computer systems, including the student system, **Eden**, and on the PC's and Mac's in computer labs. **Mathematica** is less widely available. **Matlab** is used in many engineering applications, and so engineering students will certainly become acquainted with it.

These huge programs are all examples of *computer algebra systems*. The programs can do symbolic manipulation, numerical computation, and a wide variety of graphics. Information can be exchanged easily between the various parts of the program. **Maple** "labs" (guided explorations) will be assigned as part of Math 251. Introductory information about **Maple** and local help pages are linked to the **Math 251 web page**.

A question ...

Students may ask the following: "Why do I need to learn this stuff since a computer can do it?" Certainly a computer can tell you that 25.46 **TIMES** 38.04 (multiplication) is 968.4984, but if **PLUS** had been typed instead of **TIMES** in this problem, the result displayed would be 63.50. The user intending multiplication should have enough experience to realize that 63.50 is *not* the correct answer to 25.46 **TIMES** 38.04, and enough confidence in the machine to realize that the error likely happened because of a mistake in the input, and not in the machine itself. Similarly a machine can be asked for an antiderivative of $(x^2 + 2)/(x^2 + 1)$. The answer will be $x + \arctan(x)$. But if one or another pair of parentheses (or both) are omitted, these answers result: $2x - 2/x$, $(x^3/3) + 2 \arctan(x)$, and $(x^3/3) - (2/x) + x$. This is a simple indefinite integral, and things get much more complicated with more complicated questions. Students should know the "shape" of the answer. This technology is nearly useless without good knowledge of calculus.

Workshops

In Math 151 and 152 (and sometimes in Math 251), one meeting each week takes the form of a *workshop*. In workshops students work in small groups on problems. Solutions to these problems are written up by students after class and are graded for content and clarity of exposition. Further information about workshops follows, including a sample solution and some comments. This material is also linked to the **Math 151 web page**.

Why workshops?

Math 151 and Math 152 (and sometimes Math 251) use *workshops*. This word is the Rutgers name for problem-solving in small groups during class. Students later hand in written work, called *writeups*, which are solutions to the designated problems. Writeups are graded both for mathematical accuracy and for exposition. The workshop problems are intended to be more realistic than many textbook problems, and may be more informal and less precise than textbook problems. Solutions use the methods presented in the course. Workshop problems usually will be related to the material currently being studied, but solutions may not be straightforward applications of one formula or algorithm.

Here are some reasons for workshops and writeups.

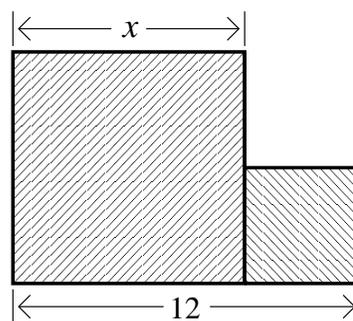
1. In the real world, academic or corporate or industrial, people most often work in teams. Team members discuss problems. Students should learn to work effectively in groups to solve problems.
2. In the real world, problems are frequently not properly and carefully stated. A major step towards solution may sometimes be formulating a precise statement of the problem! Solutions can't be obtained by looking at the models in the textbook or the answer in "the back of the book". More open-ended problems are therefore representative of this situation.
3. In the real world, a problem is rarely solved by writing a mass of algebraic manipulation and then circling "the answer" on the page. Information must be communicated supporting the suggested solution. Written communication should be clear and grammatical using complete (English!) sentences. And, in fact, neatness counts.

The next page contains a sample workshop problem followed, on the page after, by a writeup in response to this problem. You may want to think about the solution before reading the sample writeup.

An example of a workshop problem

This problem might be given after several weeks of Math 151. It is an example. The problem statement is followed by a solution. The solution is, in turn, followed by some comments which apply generally to workshops and their writeups.

Problem statement Two squares are placed so that their sides are touching, as shown. The sum of the lengths of one side of each square is 12 feet, as shown. Suppose the length of one side of one square is x feet.



a) Write a formula for $f(x)$, the sum of the total area of the two squares. What is the domain of this function when used to describe this problem? (The domain should be related to the problem statement.) Sketch a graph of $f(x)$ on its domain.

b) The square with larger area is painted red, and the square with smaller area is painted green. The cost of red paint to cover 1 square foot is \$4, and the cost of green paint to cover 1 square foot is \$10. Let $g(x)$ be the function which gives the cost of painting the squares. Describe the function $g(x)$. Sketch a graph of $g(x)$ on its domain.

Hint Read the question *carefully*. The answer will be a piecewise-defined function. A complete answer should give all relevant information

c) Where is the function $g(x)$ continuous? Where is it differentiable? Which value of x gives the least cost?

You can try the problem yourself before reading the solution.

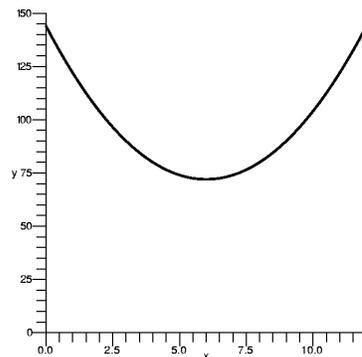
An example of a writeup for the problem

Answer to a) $f(x) = x^2 + (12 - x)^2$ and a graph of $y = f(x)$ is shown to the right.

The area of the left-hand square is x^2 . The side length of the right-hand square must be $12 - x$, so its area is $(12 - x)^2$. The total area is the sum of the two areas so

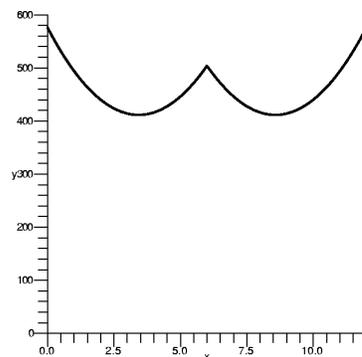
$$f(x) = x^2 + (12 - x)^2.$$

The domain is $0 < x < 12$ since squares have positive side lengths, and no side length can be greater than 12. If $x = 0$ or $x = 12$ then there would be only one square. The graph is a parabola since $x^2 + (12 - x)^2 = 2x^2 - 24x + 144$.



Answer to b) $g(x) = \begin{cases} 10x^2 + 4(12 - x)^2 & \text{if } 0 < x < 6 \\ 4x^2 + 10(12 - x)^2 & \text{if } 6 \leq x < 12 \end{cases}$ and a graph of $y = g(x)$ is shown to the right.

If $x < 6$ then the left square with side length x has smaller area. Its area is x^2 square feet, and it should be painted green at a cost of \$10 per square foot. The cost of painting this square is $10x^2$ dollars. The other square has area $(12 - x)^2$ square feet, and it should be painted red at a cost of \$4 per square foot. The larger square costs $4(12 - x)^2$ to paint. The total cost, $g(x)$, is $10x^2 + 4(12 - x)^2$ if $x < 6$.



If $x > 6$, the side length of the square with larger area is x and the side length of the square with smaller area is $12 - x$. The costs are computed as before with the multipliers for the colors reversed. The total cost, $g(x)$, is $4x^2 + 10(12 - x)^2$ if $x > 6$.

If $x = 6$, the squares have the same areas. $g(6)$ should be $10(6^2) + 4(6^2)$ regardless of how colors are assigned.

$$g(x) \text{ can therefore be defined by } g(x) = \begin{cases} 10x^2 + 4(12 - x)^2 & \text{if } 0 < x < 6 \\ 4x^2 + 10(12 - x)^2 & \text{if } 6 \leq x < 12 \end{cases}.$$

Answer to c) $g(x)$ is continuous everywhere in its domain. Since $g(x)$ is defined piecewise by polynomials, the only value of x to examine specifically is $x = 6$:

$$\lim_{x \rightarrow 6^-} g(x) = \lim_{x \rightarrow 6^-} 10x^2 + 4(12 - x)^2 = 14(6^2)$$

$$\lim_{x \rightarrow 6^+} g(x) = \lim_{x \rightarrow 6^+} 4x^2 + 10(12 - x)^2 = 14(6^2)$$

$$g(6) = 14(6^2)$$

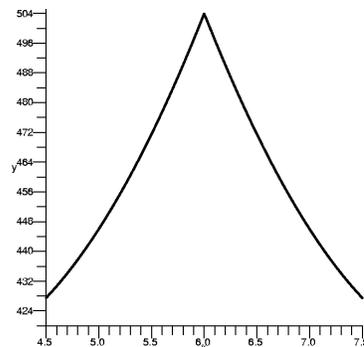
Therefore $g(x)$ is continuous at $x = 6$.

The writeup continues on the next page.

$g(x)$ is differentiable for $0 < x < 6$ and for $6 < x < 12$. $g(x)$ is not differentiable at $x = 6$.

Inside those intervals, $g(x)$ is a polynomial. Its derivative can be computed: $g'(x) = \begin{cases} 20x - 8(12 - x) & \text{if } 0 < x < 6 \\ 8x - 20(12 - x) & \text{if } 6 < x < 12 \end{cases}$.

A zoomed-in picture of $g(x)$ near $(6, g(6))$, which is the point $(6, 504)$, is shown to the right. The graph does not seem to be smooth at $x = 6$. To verify that $g(x)$ is not differentiable there consider the one-sided limits of $\frac{g(x) - g(6)}{x - 6}$ as $x \rightarrow 6^-$ and as $x \rightarrow 6^+$. If $x < 6$, then



$$\frac{g(x) - g(6)}{x - 6} = \frac{(10x^2 + 4(12 - x)^2) - (10 \cdot 6^2 + 4 \cdot 6^2)}{x - 6} = 10(x + 6) - 4(18 - x)$$

and $\lim_{x \rightarrow 6^-} 10(x + 6) - 4(18 - x) = 72$. Now the other side: if $x > 6$, then

$$\frac{g(x) - g(6)}{x - 6} = \frac{(4x^2 + 10(12 - x)^2) - (4 \cdot 6^2 + 10 \cdot 6^2)}{x - 6} = 4(x + 6) - 10(18 - x)$$

and $\lim_{x \rightarrow 6^+} 4(x + 6) - 10(18 - x) = -72$. Since $72 \neq -72$, $g(x)$ is not differentiable at $x = 6$.

The least cost is \$411.43, and this cost occurs at a critical point. $(6, 504)$ is a local maximum. The other critical points can be found by setting $g'(x) = 0$. If $0 < x < 6$, $g'(x) = 0$ means $20x - 8(12 - x) = 0$ so $x = \frac{24}{7} \approx 3.429$, and the associated cost is $g(3.429) \approx 411.429$. If $6 < x < 12$, then $g'(x) = 0$ means $8x - 20(12 - x) = 0$ so $x = \frac{60}{7} \approx 8.571$, with associated cost $g(8.571) \approx 411.429$. The costs are the same, and the lowest cost is \$411.43.

Comments about writeups follow.

Comments on the solution

Many answers ...

Calculus provides powerful tools to solve problems, and, for many problems, there may be several different correct approaches to solutions. Certainly every workshop problem will have different ways of assembling good writeups. What's given here is one example of one writeup for one workshop problem. It does *not* show how every writeup for every workshop problem should be structured.

The instructors are always available to give hints and to discuss solutions. The problems may frequently be slightly mysterious, so discussion will be useful.

Students should answer the questions that are asked, and should not invent their own questions and then answer those. Please be sure to answer the questions that are asked!

The words and the sentences

This problem has several parts, and is longer than most workshop problems. The writeup shown has 444 words (if each formula is counted as one “word”) and has 38 sentences. The prose style is straightforward, and consists mostly of short sentences: “The least cost is \$411.43, and this cost occurs at a critical point.” Other sentences give simple hypothesis/conclusion statements: “If $x > 6$, the side length of the square with larger area is x and the side length of the square with smaller area is $12 - x$.” The spelling has been checked!

Any technical communication should first be understandable. Simplicity and brevity are also virtues. One useful reference for such writing is *The Elements of Style* by Strunk and White. Old and cheap paperback versions are available. An early edition of this text is available for free on the web (the link is <http://www.bartleby.com/141/>). A summary of this text's suggestions is here: http://en.wikipedia.org/wiki/The_Elements_of_Style.

If you have taken Expository Writing, use what you learned there! It will help.

Pictures

Two pictures (graphs) were specifically requested. A third graph was given to help with explanation of the answer. Graphing calculators can display these graphs and the results can then be copied by hand to writeups. These particular graphs were done with the **Maple** program. In this case, the graphs could be obtained in elementary ways, since they are pieces of parabolas.

Clearly labeled graphs and pictures can be extremely useful parts of technical exposition. If the writeup contains a diagram describing the geometric or physical setup of a problem, label the parts of the diagram which are discussed in the writeup.

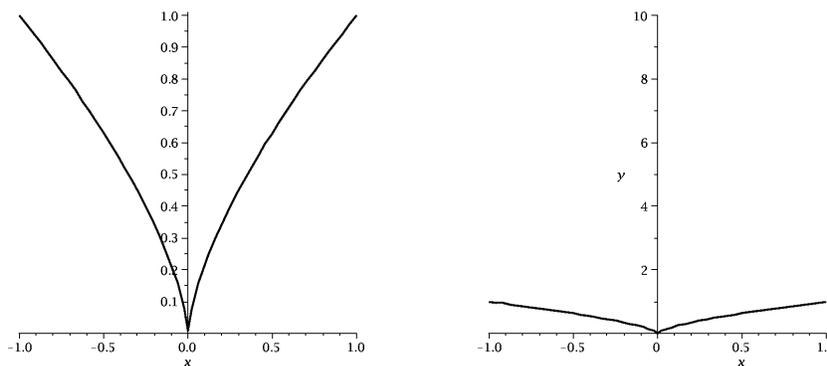
Technology

Some workshop problems will involve more numerical computations than this example and these computations should generally be carried out using a calculator or a computer. In our sample problem, the “numerics” are quite simple. The replacement of $\frac{24}{7}$ by 3.429 and of $\frac{60}{7}$ by 8.571, and the subsequent evaluations of $g(3.429)$ and $g(8.571)$ were done by a machine. The details of simple arithmetic and simple algebra are not included in the

writeup. Statements such as “ $(23.7 + 8.44) \cdot 5.3 = \text{whatever}$ ” generally should be omitted. Students may need to use calculators or computers to explore numerical results. This exploration may be part of understanding the problem statement well enough to begin solution but these explorations will usually be left out of the writeup.

The three graphs shown have windows with different aspect ratios: $\frac{12}{150}$, $\frac{12}{600}$, and $\frac{3}{84}$. Shapes are distorted, so the cusp (the corner of the graph) in the second and third graphs, which display the same curve with the same cusp, look somewhat different.

Two graphs of $x^{2/3}$ on the interval $[-1, 1]$ are below. This cusp may be more familiar. The left-hand graph has y between 0 and 1. The right-hand graph has y between 0 and 10. The aspect ratio, $\frac{\text{horizontal}}{\text{vertical}}$, changes by a factor of 5, causing the cusp to appear flatter.



Appearance

Writeups with more than one page should be stapled. Writeups should be legible. Spelling should be checked. Students should proofread their work. Neatness counts!