WHY DIAGONAL DOMINANCE IS PRESERVED IN GAUSSIAN ELIMINATION

Suppose $A = \begin{bmatrix} \vdots & \vdots & \vdots & \dots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix}$ is an $n \times n$ matrix that is **diagonally dominant** in the sense that $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for each $i = 1, \dots, n$—in words: the absolute value of the diagonal element of each row is strictly greater than the sum of the absolute values of the off-diagonal elements. Then this condition is preserved in each of the matrices produced by the Gaussian elimination algorithm on A, *without pivoting*. Indeed, the following stronger statement is true:

Proposition: Let $A = [\, a_{ij} \,]$ be a diagonally dominant $n \times n$ matrix. Define the **excess in row** i of A by

$$e_i = |a_{ii}| - \sum_{j \neq i} |a_{ij}| \ (> 0) \ .$$

Then the excess in each row of the matrix produced from A by each row step of Gaussian elimination is no smaller than the excess in that row before the step was performed.

Proof. All steps of Gaussian elimination involve the same operations as the first step, in which a multiple of the first row of A is subtracted from the second row to result in a zero entry in the $(2, 1)$ position; it will therefore suffice to show that the excess in the second row is not decreased in that operation. Consider first the case in which every diagonal element of A is 1: $a_{ii} = 1$ for all $i = 1, \dots, n$. Then $e_i = 1 - \sum_{j \neq i} |a_{ij}| > 0$ for every i and no element of the matrix is larger than 1 in absolute value. The first step of Gaussian elimination replaces

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 1 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix}$$

by

$$A^{(1)} = \begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 1 - a_{21}a_{12} & a_{23} - a_{21}a_{13} & \cdots & a_{2n} - a_{21}a_{1n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix} .$$

Both a_{21} and a_{12} were < 1 in absolute value, so the term being subtracted from 1 in the $(2, 2)$ position of $A^{(1)}$ is smaller than 1 in absolute value and thus the number $a_{22}^{(1)} = 1 - a_{12}a_{21}$ satisfies

$$|a_{22}^{(1)}| = |1 - a_{12}a_{21}| \geq 1 - |a_{12}| \cdot |a_{21}| \ .$$

(We put absolute value signs around $a_{22}^{(1)}$ because this proposition is also true when the matrices have complex entries—if we are working with real numbers, of course, this number will be positive and the absolute value signs are unnecessary.) On the other hand, the elements in the $(2, j)$ positions for $j > 2$ cannot be larger than the sum of the absolute values of their terms:

$$|a_{2j}^{(1)}| \leq |a_{2j}| + |a_{21}| \cdot |a_{1j}| \quad \text{for } j > 2.$$

Using these pessimistic estimates—taking the diagonal entry as small as it could be and the off-diagonal entries as large as they could be—we can estimate the excess $e_2^{(1)}$ in the second row of the new matrix $A^{(1)}$: noting that $a_{21} = 0$ and that by definition of e_1 we have $\sum_{j\geq 2} |a_{1j}| = 1 - e_1$, we see that

$$\begin{aligned}
e_2^{(1)} &= \left[|a_{22}^{(1)}| - \sum_{j>2} |a_{2j}^{(1)}| \right] \geq 1 - |a_{12}| \cdot |a_{21}| - \sum_{j>2} \{ |a_{2j}| + |a_{21}| \cdot |a_{1j}| \} \\
&= 1 - |a_{21}| \cdot \left[|a_{12}| + \sum_{j>2} |a_{1j}| \right] - \sum_{j>2} |a_{2j}| = 1 - |a_{21}| \cdot \left[\sum_{j\geq 2} |a_{1j}| \right] - \sum_{j>2} |a_{2j}| \\
&= 1 - |a_{21}| \cdot [1 - e_1] - \sum_{j>2} |a_{2j}| = |a_{21}|\, e_1 + \left[1 - |a_{21}| - \sum_{j>2} |a_{2j}| \right] \\
&= |a_{21}|\, e_1 + \left[1 - \sum_{j\neq 2} |a_{2j}| \right] = |a_{21}|\, e_1 + e_2 \geq e_2 \ .
\end{aligned}$$

Now suppose that it had not been the case that each diagonal element of A had been 1. We could have scaled the first and second rows of A, multiplying the first row by $1/a_{11}$ and the second row by $1/a_{22}$, and then done the computations above on the scaled matrix—call it $\tilde{A}$. This would have scaled the excesses on the corresponding rows to $\tilde{e}_1 = e_1/|a_{11}|$ and $\tilde{e}_2 = e_2/|a_{22}|$ respectively, so the excess in the second row of $\tilde{A}^{(1)}$ would really have been estimated from below by $|\tilde{a}_{21}| \cdot e_1/|a_{11}| + e_2/|a_{22}| \geq e_2/|a_{22}|$. Now the relations

$$\tilde{a}_{2j}^{(1)} = \tilde{a}_{2j} - \tilde{a}_{21}\tilde{a}_{1j} = \frac{a_{2j}}{a_{22}} - \frac{a_{21}}{a_{22}} \cdot \frac{a_{1j}}{a_{11}} = \frac{1}{a_{22}} \left(a_{2j} - a_{21}\, \frac{a_{1j}}{a_{11}} \right)$$

tell us that the second row of $A^{(1)}$ only differs from the second row of $\tilde{A}^{(1)}$ by the scaling factor $1/a_{22}$; therefore, the excesses of the second rows of $A^{(1)}$ and $\tilde{A}^{(1)}$ are related by the scale factor $1/|a_{22}|$, so $e_2^{(1)} = |a_{22}| \cdot \tilde{e}_2^{(1)} \geq |a_{22}| \cdot \tilde{e}_2 = |a_{22}| \cdot e_2/|a_{22}| = e_2$, proving the general assertion that the excess of the second row—and therefore of any row—is not diminished by a step of Gaussian elimination.