Problem Set 10.

1. Let \( l \) and \( p \) be odd primes with \( l \equiv 1 \pmod{3} \).
   a) Show that \( l \) splits completely in \( \mathbb{Q}(\sqrt[3]{p}) \) if and only if \( p \) splits completely in the cubic subfield of the cyclotomic field \( \mathbb{Q}(\zeta_l) \).
   b) Show that if 2 is a cube modulo \( l \) the ring of integers of the cubic subfield of the cyclotomic field \( \mathbb{Q}(\zeta_l) \) is not of the form \( \mathbb{Z}[\alpha] \).

2. Let \( L/K \) be a Galois extension of number fields. We say that a subextension \( F/K \) is unramified everywhere if every prime of \( \mathcal{O}_K \) is unramified in \( F \).
   a) Show that the fixed field of the subgroup \( H \) generated by all inertia groups is an extension of \( K \) which is unramified everywhere, and is the maximal such everywhere unramified extension of \( K \) contained in \( L \).
   b) Suppose that the only unramified everywhere extension of \( K \) is \( K \) itself (for example, this occurs if \( K = \mathbb{Q} \)). Show that \( \text{Gal}(L/K) \) is generated by the collection of inertia groups associated to primes of \( L \).
   c) Show that in any cyclic extension \( L \) of \( \mathbb{Q} \) of prime power degree there exists a prime \( p \) which ramifies totally (that is \( p \mathcal{O}_L \) is the \( [L : \mathbb{Q}] \) power of a prime ideal).
   d) Suppose that \( L \) is a Galois cyclic extension of \( K \) of order the square of a prime. Let \( F \) be the unique proper intermediate field. Show that any prime of \( K \) which is ramified in \( F \) ramifies totally in \( L/K \).

3. Let \( k/\mathbb{Q} \) be a quadratic field and let \( K/\mathbb{Q} \) be a Galois extension which is unramified over \( k \).
   a) Show that \( \text{Gal}(K/\mathbb{Q}) \) is generated by elements of order 2 which are not in \( \text{Gal}(K/k) \).
   b) Show that if \( \text{Gal}(K/k) \) has exponent 2 then so does \( \text{Gal}(K/\mathbb{Q}) \) and the dimension of \( \text{Gal}(K/\mathbb{Q}) \) as a vector space over the field with 2 elements is at most the number of primes ramified in \( k/\mathbb{Q} \). Conclude that there is a maximal exponent 2 unramified Galois extension of \( k \). (Hint: For the last statement, consider number fields \( E_i \) such that \( E_i \) is unramified and Galois over \( k \) and \( \text{Gal}(E/k) \) has exponent 2 and show that the compositum \( E_1 \ldots E_s \) has the same property. Let \( L \) be a number field containing \( E_i \) which is Galois over the rationals, with Galois group \( \text{Gal}(L/\mathbb{Q}) \). Let \( K \) be the compositum of the fields \( \sigma E_i \) for \( \sigma \in \text{Gal}(L/\mathbb{Q}) \). Show \( K \) is Galois over \( \mathbb{Q} \), unramified over \( k \) and \( \text{Gal}(K/k) \) has exponent 2).
   c) Show that if \( p \) is an odd rational prime which ramifies in \( k \) and \( k' \) is a quadratic extension of \( \mathbb{Q} \) ramified only at \( p \) then \( kk' \) is an unramified extension of \( k \) and \( \text{Gal}(kk'/k) \) has order at most 2.
d) Show that the discriminant of \( k \) is a product of integers \( d_j \) such that \( \mathbb{Q}(\sqrt{d_j}) \) is ramified at a single prime. Show that the maximal exponent 2 unramified Galois extension of \( k \) is \( \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_t}) \). Show that a) and b) imply that any unramified extension of \( k \) which is abelian over \( \mathbb{Q} \) is contained in this field.

Remark: The narrow genus field of a field extension \( L/F \) is the maximal unramified extension of \( L \) which is Galois and abelian over \( F \). The exercise above determines the narrow genus field of a quadratic extension of the rational field.

4. Show that given a Galois extension \( L \) of the rationals with quaternion Galois group there is a prime \( p \) which has ramification index 4. Conclude that \( L \) is ramified over every quadratic subfield.