EXERCISES II

1. Exercises from Wheeden & Zygmund, pp. 31–32: 1, 6, 10, 15, 18.

2. Mean-value Theorem #1: For $[a,b] \subseteq \mathbb{R}$, suppose $g : [a,b] \to \mathbb{R}$ is an increasing function and $f : [a,b] \to \mathbb{R}$ is integrable (in either sense) with respect to dg. Let $m = \inf f[[a,b]]$ and $M = \sup f[[a,b]]$.

(a) Show that there is some $c \in [m, M]$ for which

$$\int_a^b f(t)\, dg(t) = c \cdot [g(b) - g(a)] \; .$$

(b) Show that if $f \in \mathcal{C}[a,b]$, then there exists some $\xi \in [a,b]$ for which

$$\int_a^b f(t)\, dg(t) = f(\xi) \cdot [g(b) - g(a)] \; .$$

3. Mean-value theorem #2: Replacing the hypotheses of **1** above by the assumption that $f : [a,b] \to \mathbb{R}$ is a monotonic function and $g : [a,b] \to \mathbb{R}$ is continuous,

(a) Show that there exists $\xi \in [a,b]$ for which

$$\int_a^b f(t)\, dg(t) = f(a) \cdot [g(\xi) - g(a)] + f(b) \cdot [g(b) - g(\xi)] \; .$$

(b) Assuming further that $f(b) \cdot [f(a) - f(b)] > 0$, show that there exists $\xi \in [a,b]$ for which

$$\int_a^b f(t) g(t)\, dt = f(a) \cdot \int_a^\xi g(t)\, dt \; . \qquad \text{(Riemann integrals)}$$

[Use Wheeden & Zygmund's Prob. 15, p. 32 as a lemma.]

(c) Show (using **(b)**) that if $0 < a < b$, then $\left| \int_a^b \frac{\sin t}{t}\, dt \right| < \frac{2}{a}$.

4. Prove this minimum-hypotheses Fundamental Theorem of the Integral Calculus: If $x \mapsto F(x)$ is continuous on $[a,b]$ and differentiable on (a,b), and if $f : [a,b] \to \mathbb{R}$ is Riemann-integrable on $[a,b]$ and $f(x) = F'(x)$ for $x \in (a,b)$, then $\int_a^b f(x)\, dx = F(b) - F(a)$.