9. *The Fubini-Tonelli Theorems for (σ-)Finite Spaces.*

There are actually two versions of these theorems that we have to consider: the version for products of finitely many σ-finite measure spaces, and the version for products of infinitely many spaces, each of total mass 1 (probability spaces). The two cases have to be treated separately, since the first case acts as a set of lemmas for the second.

9.A. *Finite Products of (σ-)Finite Spaces.*

There is no particular tactical advantage in considering products of arbitrarily (but finitely) many factors; the two-factor case already contains all the technical problems that would occur with (finite) products of more than two factors, and by restricting our considerations to it we avoid drowning in a sea of indices. The setup is the one a reasonable person would expect: one has two σ-finite nonnegative measure spaces $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$, and one wants to generalize the things that happen with the natural product $\mathbb{R}^{n+m} \simeq \mathbb{R}^n \times \mathbb{R}^m$. The new set will be the Cartesian product $X \times Y$, and we shall eventually want the new σ-algebra to be the σ-algebra $\mathfrak{M} \otimes \mathfrak{N}$ generated (in $\mathbf{2}^{X\times Y}$) by all sets of the form $\{E \times F : E \in \mathfrak{M}, F \in \mathfrak{N}\}$, which we shall call **measurable rectangles** for short. We shall want each set of that form to have measure $\lambda(E \times F) = \mu(E) \cdot \nu(F)$, and the thing that has to be proved is that there is a measure (preferably a unique one, of course) satisfying that requirement on $\mathfrak{M} \otimes \mathfrak{N}$.

So we do the natural things, beginning by making the

Definition: The **product outer measure** of μ and ν on $\mathbf{2}^{X\times Y}$, which we shall denote by λ^*, is the $\overline{\mathbb{R}}^+$-valued set function on $\mathbf{2}^{X\times Y}$ defined by

$$\lambda^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \mu(E_j) \cdot \nu(F_j) : A \subseteq \bigcup_{j=1}^{\infty} E_j \times F_j,\ E_j \in \mathfrak{M},\ F_j \in \mathfrak{N} \right\} .$$

With no loss of generality in the σ-finite setup, we can always assume that this infimum is taken under the further condition that $\mu(E_j) < \infty$ and $\nu(F_j) < \infty$. It is trivial to check that this function satisfies the defining conditions for a Carathéodory outer measure, so that if the λ-measurable subsets of $X \times Y$ are defined by

> $E \subseteq X \times Y$ is **λ-measurable** if the condition $\lambda^*(A) = \lambda^*(A \cap E) + \lambda(A \setminus E)$ holds for every $A \subseteq X \times Y$,

then the general Carathéodory construction assures us that the λ-measurable sets form a σ-algebra in $\mathbf{2}^{X\times Y}$. As usual, by λ we shall mean the restriction of λ^* to the λ-measurable sets. We now should check two things: that the λ^* outer measure of measurable rectangles $E \times F$ is $\mu(E) \cdot \nu(F)$ for $E \in \mathfrak{M}$, $F \in \mathfrak{N}$, and that measurable rectangles are indeed λ-measurable.

Proposition: If $E \in \mathfrak{M}$ and $F \in \mathfrak{N}$ then $\lambda^*(E \times F) = \mu(E) \cdot \nu(F)$.

Proof. Since $E \times F$ covers itself, the definition of λ^* gives us $\lambda^*(E \times F) \le \mu(E) \cdot \nu(F)$. Suppose, on the other hand, that $E \times F \subseteq \bigcup_{j=1}^{\infty} E_j \times F_j$, where each $E_j \in \mathfrak{M}$ and each $F_j \in \mathfrak{N}$, and without loss of generality assume that all the F_j's have finite measure. This relation on the sets is logically equivalent to having the relation

$$\chi_E(x) \cdot \chi_F(y) = \chi_{E\times F}(x, y) \le \sum_{j=1}^{\infty} \chi_{E_j}(x) \cdot \chi_{F_j}(y)$$

hold at each point $(x, y) \in X \times Y$. For each $x \in X$ one can integrate both sides of this relation with respect to $\nu(y)$, obtaining (by the monotone convergence theorem in $(Y, \mathfrak{N}, \nu)$)

$$\chi_E(x) \cdot \nu(F) \le \sum_{j=1}^{\infty} \chi_{E_j}(x) \cdot \nu(F_j) .$$

This can in turn be integrated with respect to $\mu(x)$ to yield

$$\mu(E) \cdot \nu(F) \leq \sum_{j=1}^{\infty} \mu(E_j) \cdot \nu(F_j) \ .$$

Thus $\mu(E) \cdot \nu(F)$ minorizes all the numbers whose infimum defines $\lambda^*(E \times F)$, and that establishes the proposition.[36]

Proposition: Each set of the form $E \times Y$, where $E \in \mathfrak{M}$ (a "strip" or "(X-)cylinder") is λ-measurable; the same holds for sets of the form $X \times F$, where $F \in \mathfrak{N}$.

Proof. By the evident symmetry of the situation it suffices to check the first assertion. As usual in checking measurability in the Carathéodory construction, it suffices to show that for any $A \subseteq X \times Y$ with $\lambda^*(A) < \infty$, one has $\lambda^*(A) \geq \lambda^*(A \cap (E \times Y)) + \lambda^*(A \setminus (E \times Y))$. Let $\epsilon > 0$ be given and find measurable rectangles $\{E_j \times F_j\}_{j=1}^{\infty}$ for which $\displaystyle\sum_{j=1}^{\infty} \mu(E_j) \cdot \nu(F_j) < \lambda^*(A) + \epsilon$. Then each of these rectangles is the disjoint union of the two rectangles $(E_j \times F_j) \cap (E \times Y) = (E_j \cap E) \times F_j$ and $(E_j \times F_j) \setminus (E \times Y) = (E_j \setminus E) \times F_j$. Evidently the rectangles of the first form cover $A \cap (E \times Y)$ and the rectangles of the second form cover $A \setminus (E \times Y)$, so we have

$$\begin{aligned}\lambda^*(A) + \epsilon \geq \sum_{j=1}^{\infty} \mu(E_j) \cdot \nu(F_j) &= \sum_{j=1}^{\infty} \mu(E_j \cap E) \cdot \nu(F_j) + \sum_{j=1}^{\infty} \mu(E_j \setminus E) \cdot \nu(F_j) \\ &\geq \lambda^*(A \cap (E \times Y)) + \lambda^*(A \setminus (E \times Y))\end{aligned}$$

and since $\epsilon > 0$ was arbitrary, the proposition is established.

Corollary: Each measurable rectangle is λ-measurable (and its measure takes the correct value).[37] The restriction of λ to the σ-algebra $\mathfrak{M} \otimes \mathfrak{N}$ generated by the measurable rectangles is a (nonnegative, countably additive, σ-finite) measure.

Proof. The λ-measurability of the measurable rectangles follows from the relation $E \times F = (E \times Y) \cap (X \times F)$. The family of λ-measurable sets produced by the Carathéodory construction forms a σ-algebra, which must contain $\mathfrak{M} \otimes \mathfrak{N}$; so the restriction of λ to $\mathfrak{M} \otimes \mathfrak{N}$ is a countably additive measure, which obviously inherits nonnegativity and σ-finiteness from μ and ν.

It is nice to have a notation resembling that of a product for the measure $\lambda | \mathfrak{M} \otimes \mathfrak{N}$, so we shall call it $\mu \otimes \nu$; but we shall also reserve the letter λ to denote it when we want to emphasize that we do *not* want to take its origin as a product into account.

We now have a measure on $(X \times Y, \mathfrak{M} \otimes \mathfrak{N})$ that does what we want it to do, but we still have to establish the fact that integrals with respect to $\mu \otimes \nu$ can be computed as iterated integrals. In the case of the product of Lebesgue measures on $\mathbb{R}^{n+m}$ we were able to run some topological arguments that made life easier; in this situation we shall have to do some small-scale set theory.

Lemma: Let $\mathfrak{A}$ be a Boolean algebra of subsets of a set X (*i.e.*, a subfamily of $\mathbf{2}^X$ closed under union and complementation [and therefore closed under intersection and containing $\emptyset$ and X]). Let $\{A_j\}_{j=1}^{n} \subseteq \mathfrak{A}$ be a finite subset of $\mathfrak{A}$. Then the smallest subfamily of $\mathfrak{A}$ containing $\{A_j\}_{j=1}^{n}$ and closed under union, intersection and relative complementation can be expressed as the family of finite, disjoint unions of the (at most 2^n distinct) sets having the form $S_1 \cap \cdots S_n$, where for each index j either $S_j = A_j$ or $S_j = X \setminus A_j$.

This is easily proved by recalling that union is an idempotent operation and by observing that two *different* sets of the form just described intersect in the empty set (some of those sets may in fact be empty

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[36] This argument obviously generalizes to the case of a product of finitely many measure spaces without any trouble. It is unfortunate that the generalization to infinite products is not quite as transparent.

[37] This argument generalizes to infinite products with little difficulty: the intersections must be countable, but that has to be true for other reasons as well.

as defined). One is free to talk about this in terms involving finitely generated Boolean algebras being homomorphic images of finitely generated free Boolean algebras[38], if that's where one's heart lies.

Corollary: Let $\{E_j \times F_j\}_{j=1}^n$ be a finite set of rectangles with $E_j \in \mathfrak{M}$ and $F_j \in \mathfrak{N}$. Then there exist a finite disjoint subfamily $\mathfrak{E} \subseteq \mathfrak{M}$ and a finite disjoint subfamily $\mathfrak{F} \subseteq \mathfrak{N}$ such that every rectangle of the given family is contained in the family of disjoint (finite) unions of rectangles of the form $E \times F$ where $E \in \mathfrak{E}$ and $F \in \mathfrak{F}$. This family is closed under all the (finite) Boolean operations.

It suffices to find a finite family $\mathfrak{E} \subseteq \mathfrak{M}$ such that every E_j and every $X \setminus E_j$ is a disjoint union of elements of $\mathfrak{E}$ and a corresponding $\mathfrak{F} \subseteq \mathfrak{N}$ such that every F_j and every $Y \setminus F_j$ is a disjoint union of elements of $\mathfrak{F}$.

Corollary: The family of finite disjoint unions of measurable rectangles is a Boolean algebra.

Proposition: If $\{E_j \times F_j\}_{j=1}^n$ is a finite disjoint set of rectangles with $E_j \in \mathfrak{M}$ and $F_j \in \mathfrak{N}$, then the union $A = \bigcup_{j=1}^n E_j \times F_j$ has the property that for every $y \in Y$, every section $A_y = \{x \in X : (x, y) \in A\}$ belongs to $\mathfrak{M}$, and similarly every section $A^x = \{y \in Y : (x, y) \in A\}$ belongs to $\mathfrak{N}$. The functions $x \mapsto \nu(A^x) = \displaystyle\int_Y \chi_A(x, y)\, d\nu(y)$ and $y \mapsto \mu(A_y) = \displaystyle\int_X \chi_A(x, y)\, d\mu(x)$ are $\mathfrak{M}$- and $\mathfrak{N}$-measurable respectively, and

$$(\mu \otimes \nu)(A) = \int_{X\times Y} \chi_A \, d\lambda = \int_X \left[\int_Y \chi_A(x, y)\, d\nu(y) \right] d\mu(x) = \int_Y \left[\int_X \chi_A(x, y)\, d\mu(x) \right] d\nu(y) \; .$$

Proof. Indeed, the measurability of each section is immediate. The characteristic function of any such A can be written in the form $\chi_A(x, y) = \displaystyle\sum_{j=1}^n \chi_{E_j}(x) \cdot \chi_{F_j}(y)$; from the standpoint of λ this is a simple function whose integral is the sum of the measures of the rectangles, while the iterated integrals of each term are integrals of measurable functions with the value of the iterated integrals equal to the λ-measure of the corresponding rectangle. The measurablility of the sections is also obvious from the form of the chararacteristic function of A.

Corollary: The relation

$$(\mu \otimes \nu)(A) = \int_{X\times Y} \chi_A \, d\lambda = \int_X \left[\int_Y \chi_A(x, y)\, d\nu(y) \right] d\mu(x) = \int_Y \left[\int_X \chi_A(x, y)\, d\mu(x) \right] d\nu(y)$$

holds for the Boolean subalgebra of $2^{X\times Y}$ generated by the measurable rectangles.

Indeed, given any finite family of measurable rectangles we can find a finite disjoint family of disjoint measurable rectangles such that the family of all their finite unions is a Boolean algebra containing the given family. The union of these Boolean algebras is thus the Boolean algebra generated by the measurable rectangles, and we have the equality of these integrals on any of those families of finite unions of disjoint families.

Establishing the measurability of the sections A^x and A_y does not require anything like what we just developed: for each $x \in X$ the family of sets $\{A \subseteq \mathbf{2}^{X\times Y} : A_y \in \mathfrak{M} \text{ for every } y \in Y\}$ is easily seen to be a σ-algebra (because $\mathfrak{M}$ is) that contains every measurable rectangle and must therefore contain $\mathfrak{M} \otimes \mathfrak{N}$, and similarly for the A^x's. However, the measurability of the functions $x \mapsto \nu(A^x) = \int_Y \chi_A(x, y)\, d\nu(y)$ and $y \mapsto \mu(A_y) = \int_X \chi_A(x, y)\, d\mu(x)$ is not as obvious: the monotone convergence theorem lets us take increasing limits always and decreasing limits under a finiteness condition, but for a general $A \in \mathfrak{M} \otimes \mathfrak{N}$ it is not clear where to start. We need a new notion in order to extend these results to apply to all $A \in \mathfrak{M} \otimes \mathfrak{N}$, and it is furnished by the following

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[38] See, *e.g.*, G. Birkhoff, *Lattice Theory*, rev. ed., Amer. Math. Soc. (1961), p. 163 ff.

Definition: A family $\mathfrak{K}$ of subsets of a given set S is a **monotone class** if it is closed under taking monotone sequential limits, *i.e.*, if $\{S_j\}_{j=1}^{\infty} \subseteq \mathfrak{K}$ and $S_1 \supseteq S_2 \supseteq \cdots S_j \supseteq S_{j+1} \supseteq \cdots$ implies $\displaystyle\bigcap_{j=1}^{\infty} S_j \in \mathfrak{K}$ and $\{S_j\}_{j=1}^{\infty} \subseteq \mathfrak{K}$ and $S_1 \subseteq S_2 \subseteq \cdots S_j \subseteq S_{j+1} \subseteq \cdots$ implies $\displaystyle\bigcup_{j=1}^{\infty} S_j \in \mathfrak{K}$.

It is obvious that the intersection of an arbitrary family of monotone classes in $\mathbf{2}^S$ is a monotone class, so every family $\mathfrak{A} \subseteq \mathbf{2}^S$ is contained in a smallest monotone class, namely the intersection of all the monotone classes containing it, and this is called the **monotone class generated by** $\mathfrak{A}$.[39] For some families, the two notions coincide.

Proposition: If $\mathfrak{A}$ is a Boolean algebra, then the monotone class generated by $\mathfrak{A}$ equals the σ-algebra generated by $\mathfrak{A}$.

Proof. A σ-algebra is a monotone class, so the monotone class generated by $\mathfrak{A}$ is contained in the σ-algebra generated by $\mathfrak{A}$. The monotone class generated by $\mathfrak{A}$ is also closed under complementation, since the family of complements of a given monotone class is also a monotone class and if a monotone class contains $\mathfrak{A}$, then the family of its complements also contains $\mathfrak{A}$. On the other hand, for any $A \subseteq B$ consider the family $\mathfrak{K}(A)$ consisting of all sets $B \subseteq S$ for which the sets $A \cup B$, $A \cap B$, $A \setminus B$ and $B \setminus A$ belong to the monotone class generated by $\mathfrak{A}$. For fixed A, the family $\mathfrak{K}(A)$ is itself a monotone class, and $A \in \mathfrak{K}(B)$ if and only if $B \in \mathfrak{K}(A)$. If we start with $A \in \mathfrak{A}$, then $\mathfrak{K}(A) \supseteq \mathfrak{A}$ and therefore $\mathfrak{K}(A)$ contains the monotone class generated by $\mathfrak{A}$. It follows that if B belongs to the monotone class generated by $\mathfrak{A}$ then $B \in \mathfrak{K}(A)$ and therefore $A \in \mathfrak{K}(B)$. But now $\mathfrak{K}(B) \supseteq \mathfrak{A}$ and thus $\mathfrak{K}(B)$ contains the monotone class generated by $\mathfrak{A}$. This says: for any sets A and B belonging to the monotone class generated by $\mathfrak{A}$, the sets $A \cup B$, $A \cap B$, $A \setminus B$ and $B \setminus A$ again belong to the monotone class generated by $\mathfrak{A}$. But now we see that the monotone class generated by $\mathfrak{A}$ is closed under the finite set-theoretic operations, and since it is closed under monotone sequential union and intersection, it is closed under countable union and intersection. Thus it is a σ-algebra containing $\mathfrak{A}$, and so it contains and thus equals the σ-algebra generated by $\mathfrak{A}$.

We now apply that proposition to the family of sets $A \subseteq X \times Y$ that satisfy the following conditions:

(1) A is λ-measurable;

(2) Every section $A_y \in \mathfrak{M}$ (where $y \in Y$) and every section $A^x \in \mathfrak{N}$ (where $x \in X$);

(3) For every $E \in \mathfrak{M}$ and $F \in \mathfrak{N}$ with $\mu(E) < \infty$ and $\nu(F) < \infty$, the functions $x \mapsto \nu(A^x \cap F) = \displaystyle\int_F \chi_A(x,y)\,d\nu(y)$ and $y \mapsto \mu(A_y \cap E) = \displaystyle\int_E \chi_A(x,y)\,d\mu(x)$ are $\mathfrak{M}$- and $\mathfrak{N}$-measurable $\mathbb{R}^+$-valued functions respectively;

(4) For every $E \in \mathfrak{M}$ and $F \in \mathfrak{N}$ with $\mu(E) < \infty$ and $\nu(F) < \infty$, the relation

$$(\mu\otimes\nu)(A\cap(E\times F)) = \int_{E\times F} \chi_A\,d\lambda = \int_E \left[\int_F \chi_A(x,y)\,d\nu(y)\right] d\mu(x) = \int_F \left[\int_E \chi_A(x,y)\,d\mu(x)\right] d\nu(y)$$

holds.

The monotone convergence theorem assures us that this family is a monotone class: membership is preserved both for monotone increasing and for monotone decreasing sequences because the integrals are taken over sets of finite measure, and the inner integrals define bounded functions. We know that the family contains the Boolean algebra of finite unions of measurable rectangles, so the proposition proved immediately above shows that it contains the σ-algebra $\mathfrak{M} \otimes \mathfrak{N}$ generated by them, and we have all but proved the

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[39] This is, of course, the same kind of impredicative definition that produces the σ-algebra generated by a given family.

Theorem: Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two σ-finite measure spaces, let $\mathfrak{M} \otimes \mathfrak{N}$ be the σ-algebra in $\mathbf{2}^{X \times Y}$ generated by all sets of the form $E \times F$ where $E \in \mathfrak{M}$ and $F \in \mathfrak{N}$, and let λ be the measure defined on $\mathfrak{M} \otimes \mathfrak{N}$ by restricting to this σ-algebra the Carathéodory measure obtained from the outer measure

$$\lambda^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu(E_j) \cdot \nu(F_j) : A \subseteq \bigcup_{j=1}^{\infty} E_j \times F_j,\ E_j \in \mathfrak{M},\ F_j \in \mathfrak{N} \right\}$$

by the Carathéodory extension process. Then for every $A \in \mathfrak{M} \otimes \mathfrak{N}$:

(1) A is λ-measurable;

(2) Every section $A_y \in \mathfrak{M}$ (where $y \in Y$) and every section $A^x \in \mathfrak{N}$ (where $x \in X$);

(3) The functions $x \mapsto \nu(A^x) = \int_Y \chi_A(x,y)\, d\nu(y)$ and $y \mapsto \mu(A_y) = \int_X \chi_A(x,y)\, d\mu(x)$ are $\mathfrak{M}$- and $\mathfrak{N}$-measurable $\overline{\mathbb{R}}^+$-valued functions respectively;

(4) $\lambda(A) = \int_{X \times Y} \chi_A \, d\lambda = \int_X \left[\int_Y \chi_A(x,y)\, d\nu(y) \right] d\mu(x) = \int_Y \left[\int_X \chi_A(x,y)\, d\mu(x) \right] d\nu(y).$

Proof. We already know that (3) and (4) hold if one integrates only over E, F or $E \times F$ respectively, where $\mu(E) < \infty$ and $\nu(F) < \infty$. Since the measures are σ-finite, there are increasing sequences $\{E_j\}_{j=1}^{\infty} \subseteq \mathfrak{M}$ and $\{F_k\}_{k=1}^{\infty} \subseteq \mathfrak{N}$ whose unions are X and Y respectively. Applying the (increasing) monotone convergence theorem to the facts that

(3) The functions $x \mapsto \nu(A^x \cap F_k) = \int_{F_k} \chi_A(x,y)\, d\nu(y)$ and $y \mapsto \mu(A_y \cap E_j) = \int_{E_j} \chi_A(x,y)\, d\mu(x)$ are $\mathfrak{M}$- and $\mathfrak{N}$-measurable $\mathbb{R}^+$-valued functions respectively;

(4) The relation

$$\lambda(A \cap (E_j \times F_k)) = \int_{E_j \times F_k} \chi_A \, d\lambda = \int_{E_j} \left[\int_{F_k} \chi_A(x,y)\, d\nu(y) \right] d\mu(x) = \int_{F_k} \left[\int_{E_j} \chi_A(x,y)\, d\mu(x) \right] d\nu(y)$$

holds,

taking the limit first on the index pertaining to the inner integral, we have assertions (3) and (4) of the theorem.

Integration and iterated integration with respect to $\mu \otimes \nu$, μ and ν now follow about the same pattern as they do in the $\mathbb{R}^{n+m} \simeq \mathbb{R}^n \times \mathbb{R}^m$ setting. If $f : X \times Y \to \overline{\mathbb{R}}^+$ is a $\mathfrak{M} \otimes \mathfrak{N}$-measurable function, then there exist increasing sequences of $\mathfrak{M} \otimes \mathfrak{N}$-measurable ($\mathbb{R}^+$-valued) simple functions converging everywhere on $X \times Y$ to f; let $\{s_k\}_{k=1}^{\infty}$ be such a sequence. Then application of the monotone convergence theorem shows that the functions

$$\begin{aligned} x &\mapsto \int_Y f(x,y)\, d\nu(y) = \lim_{k \to \infty} \int_Y s_k(x,y)\, d\nu(y) \quad \text{and} \\ y &\mapsto \int_X f(x,y)\, d\mu(x) = \lim_{k \to \infty} \int_X s_k(x,y)\, d\mu(x) \end{aligned}$$

are $\mathfrak{M}$- and $\mathfrak{N}$-measurable $\mathbb{R}^+$-valued functions respectively, and then that we have equality of the three integrals

$$\int_{X \times Y} f(x,y)\, d(\mu \otimes \nu) = \int_X \left[\int_Y f(x,y)\, d\nu(y) \right] d\mu(x) = \int_Y \left[\int_X f(x,y)\, d\mu(x) \right] d\nu(y)$$

whether the values of the integrals be finite or $+\infty$; this is **Tonelli's Theorem** in the case of a product of σ-finite abstract measure spaces. If $f : X \times Y \to \overline{\mathbb{R}}$ is a $\mathfrak{M} \otimes \mathfrak{N}$-measurable function that belongs to

$\mathcal{L}^1(X \times Y, \mathfrak{M} \otimes \mathfrak{N}, d(\mu \otimes \nu))$, then by applying Tonelli's theorem to its positive and negative parts separately we find that the functions

$$x \mapsto \int_Y f(x,y)\, d\nu(y) \quad \text{and}$$
$$y \mapsto \int_X f(x,y)\, d\mu(x)$$

are defined on the complement of an $\mathfrak{M}$-measurable set of μ-measure zero and the complement of an $\mathfrak{N}$-measurable set of ν-measure zero, respectively (namely, in the first case the set of $x \in X$ [possibly empty] on which both $\int_Y f^+(x,y)\, d\nu(y)$ and $\int_Y f^-(x,y)\, d\nu(y)$ take the value $+\infty$, and similarly in the second case). Conventionally setting the value of these inner integrals equal to zero on the sets on which they are not defined, we can combine the integrals for f^+ and f^- to yield, first, that the inner integrals are (a. e. finite and) elements of $\mathcal{L}^1(X, \mathfrak{M}, d\mu)$ and $\mathcal{L}^1(Y, \mathfrak{N}, d\nu)$ respectively, and then the equality of the three integrals

$$\int_{X \times Y} f(x,y)\, d(\mu \otimes \nu) = \int_X \left[\int_Y f(x,y)\, d\nu(y)\right] d\mu(x) = \int_Y \left[\int_X f(x,y)\, d\mu(x)\right] d\nu(y)$$

which is the **Fubini theorem** for products of σ-finite abstract measure spaces.

In the treatment above we took some care to insure that things were measurable with respect to $\mathfrak{M}$, $\mathfrak{N}$, and $\mathfrak{M} \otimes \mathfrak{N}$, and not to ignore null sets unless they were known to be measurable. If the measure spaces $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ are **complete** in the sense that every subset of X that differs by a μ-null set from an element of $\mathfrak{M}$ also belongs to $\mathfrak{M}$, and similarly for $(Y, \mathfrak{N}, \nu)$, then it is worthwhile to enlarge $\mathfrak{M} \otimes \mathfrak{N}$ to a larger σ-algebra $\overline{\mathfrak{M} \otimes \mathfrak{N}}$ and λ to a measure $\overline{\lambda}$ on the larger algebra, such that the measure space $(X \times Y, \overline{\mathfrak{M} \otimes \mathfrak{N}}, \overline{\lambda})$ is also complete. We shall indicate the method of proof but omit much of the detail-checking, because it closely parallels what one has to do in the $\mathbb{R}^{n+m} \simeq \mathbb{R}^n \times \mathbb{R}^m$ situation: in the case of σ-finite spaces, there are no unpleasant surprises. For $\overline{\mathfrak{M} \otimes \mathfrak{N}}$ we simply take all the sets that are measurable with respect to the outer measure λ^*, the "λ-measurable sets" of p. 52 above; then $\overline{\lambda}$ is just the restriction of the outer measure λ^* to these sets. Since the measures μ and ν are σ-finite, there are increasing sequences $\{A_k\}_{k=1}^\infty \subseteq \mathfrak{M}$ and $\{B_k\}_{k=1}^\infty \subseteq \mathfrak{N}$ of sets of finite μ- and ν-measure respectively whose unions are X and Y respectively. If $S \subseteq X \times Y$ satisfies the Carathéodory condition for λ^* then so does each $S \cap (A_k \times B_k)$, so by finding a decreasing sequence of countable unions $\{U_{k,j}\}_{j=1}^\infty$ of rectangles with $S \cap (A_k \times B_k) \subseteq U_{k,j}$ and $\lambda(U_{k,j})$ (finite) decreasing to $\lambda^*(S \cap (A_k \times B_k))$, we can find sets $\{R_k\}_{k=1}^\infty \subseteq \mathfrak{M} \otimes \mathfrak{N}$ with $S \cap (A_k \times B_k) \subseteq R_k$ and $\lambda^*(R_k \setminus S_k) = 0$. If we put $H = \bigcup_{k=1}^\infty R_k$, then clearly $S \subseteq H$ and the difference $Z = H \setminus S$ of the two sets is $\overline{\lambda}$-null. While Z may not belong to $\mathfrak{M} \otimes \mathfrak{N}$, there is a set $N \in \mathfrak{M} \otimes \mathfrak{N}$ with $Z \subseteq N$ and $\lambda(N) = 0$, obtainable by the same covering argument that gave us H. We have $\chi_H(x,y) = \chi_S(x,y) + \chi_Z(x,y)$ and $0 \leq \chi_Z(x,y) \leq \chi_N(x,y)$ throughout $X \times Y$. By what we already know about iterated integrals in $(X \times Y, \mathfrak{M} \otimes \mathfrak{N}, \mu \otimes \nu)$, μ-almost every section N^x of N will be a ν-null set, and for these values of x the set Z^x (under the assumption that ν is complete) will be ν-null and therefore ν-measurable. Each such section—and thus μ-almost every section—of Z being ν-null and ν-measurable, it follows that μ-almost every section of S will be ν-measurable. The same applies with the rôles of the two measure spaces interchanged. Thus the inner integral on the r. h. s. of the relation

$$\overline{\lambda}(S) = \int_{X \times Y} \chi_S(x,y)\, d\overline{\lambda} = \int_X [\chi_S(x,y)\, d\nu(y)]\, d\mu(x)$$

is defined for μ-almost all x, the set of non-definition is μ-measurable under the assumption that $(X, \mathfrak{M}, \mu)$ is complete, and it is routine to verify that the relation does in fact hold. The same applies with the rôles of the two measure spaces interchanged. We now have the "Fubini-Tonelli relations" for the characteristic functions of $\overline{\mathfrak{M} \otimes \mathfrak{N}}$-measurable sets, and the arguments that establish the Tonelli and Fubini theorems with the inner integrals defined only on the complements of sets of measure zero can now be transcribed from §§3–4 of the notes on "Lemmas for Wheeden & Zygmund's Chapter 6" in a routine, almost word-for-word fashion. The material of §5 on distribution functions and the Fubini-Tonelli theorems works equally well in this context. So when one takes the product of two complete σ-finite measure spaces, one can produce a complete(d) product $(X \times Y, \overline{\mathfrak{M} \otimes \mathfrak{N}}, \mu \overline{\otimes} \nu)$ if one wishes. Indeed, it is necessary to do this if one wants an "abstract" measure theory that will give the product of Lebesgue measures on $\mathbb{R}^n$ and $\mathbb{R}^m$ as Lebesgue measure on $\mathbb{R}^{n+m}$—Lebesgue measure is complete.