501 Lebesgue-Stieltjes measures

Let F be a right-continuous, non-decreasing, real valued function on $\mathbb{R}$. For each left-half open interval $(a, b]$, define

$$\overset{\circ}{m}_F \; ((a, b]) \stackrel{\triangle}{=} F(b) - F(a)$$

For each finite disjoint union $A = \cup_1^n (a_i, b_i]$, define

$$\overset{\circ}{m}_F \; (A) = \sum_1^n F(b_i) - F(a_i).$$

Then $\overset{\circ}{m}_F$ defines a finitely additive measure on the algebra $\mathcal{V}$ of finite disjoint unions of left-half open intervals. It is a generalization of the length measure $\overset{\circ}{m}$ defined on $\mathbb{R}$ in the construction of Lebesgue measure; if F is the identity function $\overset{\circ}{m}_F \; ((a, b]) = b - a$. The proof that $\overset{\circ}{m}_F$ is consistently defined and is finitely additive goes just like the proof for $\overset{\circ}{m}$ and is omitted.

Proposition 1 $\overset{\circ}{m}_F$ *is continuous from below on* $(\mathbb{R}, \mathcal{V})$.

Proof: The proof here follows the same strategy as the proof of the continuity from below of $\overset{\circ}{m}$ as given in section 5.1 of the lecture notes on *Construction of Measures*, so we only put in enough details to show where the assumption that F is right-continuous is used.

For the same reason as was argued for $\overset{\circ}{m}$, it suffices to show that $\overset{\circ}{m}_F$ is continuous from above at $\emptyset$, and this is what we shall prove. The crucial step in the proof for $\overset{\circ}{m}$ was to show that for each bounded B in $\mathcal{V}$ and for any $\epsilon > 0$, there exists a B' in $\mathcal{V}$ so that the closure $\bar{B}'$ of B' is contained in B and $\overset{\circ}{m} \; (B - B') < \epsilon$. The right-continuity of F allows the generalization of this fact to $\overset{\circ}{m}_F$. To see this denote the representation of of a bounded set B in $\mathcal{V}$ as a finite disjoint union be $B = \cup_1^n (a_i, b_i]$. For $\delta > 0$, define $B_\delta \stackrel{\triangle}{=} \cup_1^n (a_i + \delta, b_i]$. Observe that for all δ sufficiently small,

$$\overset{\circ}{m}_F \; (B - B_\delta) = \sum_1^n F(a_i + \delta) - F(a_i).$$

Therefore, given any $\epsilon > 0$, the right-continuity of F implies that there is a positive δ such that $\overset{\circ}{m}_F \; (B - B_\delta) < \epsilon$. Clearly, $\bar{B}_\delta \subset B$.

The proof of continuity from above at $\emptyset$ can now proceed as before. Assume that $\{A_n\}$ is a sequence in $\mathcal{V}$, that $A_n \downarrow \emptyset$ and that $\overset{\circ}{m}_F \; (A_1) < \infty$, so that A_1, and hence all A_n are bounded. To avoid triviality, assume also that $\overset{\circ}{m}_F \; (A_n) > 0$ for all n.

Fix an $\epsilon > 0$, arbitrary For each n, let A'_n be a set in $\mathcal{V}$ such that $\bar{A}'_n \subset A_n$ and $\overset{\circ}{m}_F\,(A_n - A'_n) < \epsilon/2^n$. Let $F_n = \cap_1^n A'_i$. Then, as before $\overset{\circ}{m}_F\,(A_n - F_n) < \epsilon$ and $\bar{F}_n \downarrow \emptyset$. The set F_n are compact, and therefore there is an N so that F_N is empty, and $m_F(A_n) \leq \overset{\circ}{m}_F\,(A_N) < \epsilon$ for all $n \geq N$. Since ϵ was arbitrary, we have shown $\overset{\circ}{m}_F\,(A_n) \downarrow 0$.

By the extension theorem, $\overset{\circ}{m}_F$ admits a unique extension to a measure m_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, called the Lebesgue-Stieltjes measure. Of course, this can be completed to get the measure space is $(\mathbb{R}, \mathcal{M}_F, m_F)$, where $\mathcal{M}_F$ is the σ-algebra of m_F^*-measurable sets, associated to the outer measure m_F^* induced by $\overset{\circ}{m}_F$. In general, $\mathcal{M}_F$ differs from the σ-algebra of Lebesgue measurable sets. For example if $F(x)$ is the function $\chi_{[0,\infty)}(x)$, the indicator function of $[0, \infty)$, then $\mathcal{M}_F$ is in fact the collection of all subsets of $\mathbb{R}$, and $m_F(A) = 1$ if $0 \in A$, $m_F(A) = 0$, if not.

Any measure on the Borel sets of $\mathbb{R}$ which is finite on every compact subset is actually a Lebesgue-Stieltjes measure. Indeed, let ρ be a such a measure. Define

$$F(x) = \begin{cases} \rho\left([0, x]\right), & \text{if } x \geq 0; \\ -\rho\left((x, 0)\right), & \text{if } x < 0. \end{cases}$$

The continuity of ρ from above and below then imply the F is right-continuous, while the monotonicity of ρ implies that F is increasing. With this definition

$$\rho\left((a, b]\right) = F(b) - F(a).$$

Hence $\rho = m_F$.

Lebesgue-Stieltjes measures are important in probability theory. A right-continuous, increasing function such that $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$ is called a *probability distribution function*, or sometimes a *cumulative distribution function*. We think of such an F as a description for an experiment that produces a random real number, in which the probability that the chosen number falls in $(a, b]$ is $F(b) - F(a)$. The requirements that $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$ are imposed so that the probability that the number falls in $(-\infty, \infty)$ is 1, as it should be. F induces the measure m_F on the Borel sets of $\mathbb{R}$ which is a probability measure in the sense that $m_F(\mathbb{R}) = 1$; in probability theory, $m_F(U)$, for a Borel set U, is then interpreted as the probability that the randomly produced number falls in U. Conversely if, P is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$,

$$F(x) \stackrel{\triangle}{=} P\left((-\infty, x]\right)$$

defines a probability distribution function such that $P = m_F$.