501 Lecture Notes: Approximation of measurable sets

Notation: Lebesgue measure on $\mathbb{R}^d$ is denoted m_d. Lebesgue outer measure on $\mathbb{R}^d$ is denoted m_d^*. Let $\mathcal{H}_d$ be the collection of left-half open rectangles in $\mathcal{R}^d$, that is, rectangles of the form

$$(\alpha_1, \beta_1] \times \cdots \times (\alpha_d, \beta_d].$$

In class we stated and proved that a Lebesgue measurable set can be approximated to within a zero measure set by a G_δ set from above and an F_σ set from below, and that, in fact, this property characterizes Lebesgue measurable sets.

Theorem 1 *The following statements are equivalent.*
(i) A is a Lebesgue measurable subset of $\mathbb{R}^d$.
(ii) There is a G_δ set U such that $A \subseteq U$, and $U \setminus A$ has zero Lebesgue measure.
(iii) There is an F_σ set K such that $K \subseteq A$ and $A \setminus K$ has zero Lebesgue measure.

These notes comment on this theorem and its proof.

For the proof we gave in class, the crucial underlying fact is the following.

Lemma 1 *If A is a Lebesgue measurable subset of $\mathbb{R}^d$, then for each $\epsilon > 0$ there is an open set G containing A such that*

$$m_d^*(G \setminus A) = m_d(G \setminus A) < \epsilon. \tag{1}$$

Notice that Wheeden and Zygmund *define* a set A to be Lebesgue measurable precisely if for each $\epsilon > 0$ there exists an open set G containing A such that $\mu^*(G - A) < \epsilon$. We have chosen rather to use Caratheodory's definition of measurable set and then to derive Lemma 1. However, once this is accomplished, the proof we gave is essentially that of Wheeden and Zygmund, Theorem 3.28, page 42. Please refer to the text for the details.

I want to point out here that Theorem 1 and its proof generalize easily. We have seen an abstract version of this generalization in problem 44. But if we specialize to measures on the Borel σ-algebra , $\mathcal{B}(\mathbb{R}^d)$, of $\mathbb{R}^d$, we get a result very similar to Theorem 1.

We shall say that a measure ν is a *Borel measure* if its a measure on the Borel σ-algebra of $\mathbb{R}^d$ for some $d \geq 1$ and if $\nu(B) < \infty$ for every *bounded* Borel set. Lebesgue-Stieltjes measures on $\mathbb{R}$ are examples of Borel measures. Recall that to define a Lebesgue-Stieltjes measure we started with a non-decreasing, right-continuous function $F : \mathbb{R} \to \mathbb{R}$, and defined the finitely additive measure $\overset{\circ}{m}_F$ on the algebra of disjoint unions of left-half open intervals so that $\overset{\circ}{m}_F((a, b]) = F(b) - F(a)$. The Lebesgue-Stieltjes measure m_F is the extension of $\overset{\circ}{m}_F$ to the collection of m_f^* measurable sets, which includes the Borel σ-algebra.

Consider a Borel measure ν on $\mathbb{R}^d$. Let $\mathcal{R}$ be the algebra of finite disjoint unions of left-half open rectangles, as usual. Let $\mathring{\nu}$ be the restriction of ν to this algebra, and let ν^* be the outer measure induced by $\mathring{\nu}$ on $\mathcal{R}$. Call the measurable sets associated to the outer measure ν^* the ν-measurable sets. Clearly, by the Carathéodory extension theorem, ν is the unique extension of $\mathring{\nu}$ to the Borel sets, and ν in turn extends to a measure on the ν-measurable sets. We continue to label this last extension by ν.

Theorem 2 *Let ν be a Borel measure on $\mathbb{R}^d$. The following statements are equivalent.*
(i) A is a ν-measurable subset of $\mathbb{R}^d$.
(ii) There is a G_δ set U such that $A \subseteq U$, and $\nu^*(U \setminus A) = 0$.
(iii) There is an F_σ set K such that $K \subseteq A$ and $\nu^*(A \setminus K) = 0$.

Of course, this theorem subsumes Theorem 1. Moreover, its proof is really the same We discuss an outline; the student should fill in the details as an exercise.

First, it is easy to establish that (ii) or (iii) imply (i) since G_δ and F_σ are Borel sets.

Statement (iii) can be proved from statement (ii) by taking complements.

To prove (ii) assuming (i), we establish:

Lemma 2 *If A is a ν-measurable subset of $\mathbb{R}^d$, then for each $\epsilon > 0$ there is an open set G containing A such that*

$$\nu^*(G \setminus A)\,(= \nu(G \setminus A)) < \epsilon. \tag{2}$$

First assume that A is bounded. Pick an $\epsilon > 0$. By definition

$$\nu(A) = \nu^*(A) = \inf\left\{\sum_1^\infty \nu(E_i)\,;\; A \subseteq \bigcup E_i, E_i \in \mathcal{H}_d\,\forall i\right\}.$$

Since A is bounded and so has finite ν-measure (by our definition of Borel measure), we may find a cover of A using bounded rectangles in $\mathcal{H}_d$, such that

$$\sum_1^\infty \nu(E_i) < \nu(A) + \epsilon/2,$$

Each E_i can be represented as an intersection of a decreasing sequence of bounded open rectangles. By continuity from above (the rectangels are bounded and so have finite ν measure), there is thus an open rectangle U_i containing E_i with $\nu(U_i) < \nu(E_i) + \epsilon/2^{i+1}$. Let $G = \bigcup U_i$. Then, since A is ν-measurable,

$$\nu^*(G \setminus A) = \nu(G) - \nu(A) < \epsilon.$$

If A is not bounded, let $\mathbb{R}^d = \bigcup_1^\infty I_n$ be a disjoint partition of $\mathbb{R}^d$ into bounded Borel sets. For each n, choose an open U_n containing $A \cap I_n$ such that $\nu(U_n \setminus A) < \epsilon/2^n$. Then, take $G = \bigcup_1^\infty U_n$.

With this proof of the lemma completed, it is easy to finish the proof that statement (i) of Theorem 2 implies statement (ii). For each n let G_n be an open set that contains A and such that $\nu(G_n - \backslash A) < 1/n$. Define the G_δ set $U = \bigcap_1^\infty G_n$. Then $\nu(G \setminus A) = 0$.

Theorem 2 shows that the difference between the class of Lebesgue measurable sets and the class of ν-measurable sets for a Borel measure is really determined by the difference between the sets of Lebesgue measure zero and the sets of ν-measure zero.