Problems, Math 501, Fall 2003

27. In each case decide whether the collection of sets $\mathcal{A}$ is a σ-algebra, an algebra only, or neither.

(i) $\mathcal{A}$ is the collection of all finite and co-finite subsets of an infinite set $\mathcal{S}$. (A subset B of S is co-finite if B^c is finite.)

(ii) $\mathcal{A}$ is the collection of all countable (i.e. finite or countably infinite) and co-countable subsets of an infinite set $\mathcal{S}$.

(iii) All open and closed subsets of $\mathbb{R}$.

28. Let $\mathcal{E}$ be a non-empty collection of subsets of a set $\mathcal{S}$, and suppose that $\mathcal{E}$ satisfies

i). $\mathcal{S} \in \mathcal{E}$.

ii). If A and B are in $\mathcal{E}$, so is $A \cap B$.

iii). If A is in $\mathcal{E}$, then A^c is a finite disjoint union of elements in $\mathcal{E}$.

Show that the set of all finite disjoint unions of elements of $\mathcal{E}$ is an algebra.

29. Let $\mathcal{A}_0$ be a non-empty collection of subsets of $\mathcal{S}$. Take $\mathcal{A}_1$ to be the set of all finite intersections of the form $\cap_1^n B_i$ where for each i, either $B_i \in \mathcal{A}$ or $B_i^c \in \mathcal{A}$. Let $\mathcal{A}_2$ be the collection of all finite disjoint unions of elements of $\mathcal{A}_\infty$. Show that $\mathcal{A}_2$ is the smallest algebra containing $\mathcal{A}_0$.

30. (Folland, chapter 1, problem 3) Let $\mathcal{F}$ be an infinite σ-algebra.

(a) Show that $\mathcal{F}$ contains an infinite sequence of non-empty disjoint sets.

(b) Show that the cardinality of $\mathcal{F}$ is at least as large as the cardinality of the real numbers.

31. Let $\Omega \triangleq \{0, , 1\}^\infty$ be the space of all sequences $\omega = (\omega_1, \omega_2, \ldots)$ of 0,s and 1's. Let $\mathcal{F}$ be the σ-algebra generated by the cylinder sets of Ω; a cylinder set is a subset of the form $\{\omega\,;\,(\omega_1, \ldots, \omega_n) \in B\}$, where n is a positive integer and B is a subset of $\{0, 1\}^n$. Show that the set

$$\left\{\omega\,;\, \lim_{n\to\infty} \frac{1}{n}\sum_1^n \omega_i = \frac{1}{2}\right\}$$

is in $\mathcal{F}$.

32 A nonempty collection $\mathcal{M}$ of subsets of $\mathcal{S}$ is called a *monotone class* if

(i) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ and if $A_i \in \mathcal{M}$ for each i, then $\cup_1^\infty A_i \in \mathcal{M}$.

(ii) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ and if $A_i \in \mathcal{M}$ for each i, then $\cap_1^\infty A_i \in \mathcal{M}$.

Prove the *Monotone Class Theorem*:

Let $\mathcal{C}$ be an algebra and suppose $\mathcal{M}$ is a monotone class containing $\mathcal{C}$. Then $\sigma(\mathcal{C}) \subseteq \mathcal{M}$

(Hint: Show that if $\mathcal{N}$ is the smallest monotone class containing $\mathcal{C}$ then $\mathcal{N} = \mathcal{C}$. Observe that it is enough to prove that $\mathcal{N}$ is a field. (Why?) This can be done by the type of method used to prove the π-λ theorem.)