501 Problems 67-

All functions are measurable with respect to the appropriate σ-algebra, unless specified otherwise.

67. (Wheeden and Zygmund, pg. 62, problem 12) If a function f defined on $\mathbb{R}$ is continuous almost everywhere, then it is Lebesgue measurable.

68. (variation on Wheeden and Zygmund, pg. 62, problem 15) Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable subset E of the measure space (S, Σ, μ). Assume that $\mu(E) < \infty$. Suppose that for each $x \in E$, $\sup_k |f_k(x)| < \infty$. Show: for each $\epsilon > 0$, there exists a measurable set $F_\epsilon \subseteq E$ and such that $\mu(E - F_\epsilon) < \epsilon$ and $\sup\{f_k(x)\,;\, k \in \mathbb{N},\, x \in F_\epsilon\} < \infty$. Hint: Take as inspiration the proof of Egorov's theorem.

69. (Wheeden and Zygmund, pg. 63, problem 17) Let $f_n \to f$ and $g_n \to g$ in measure. Show that $f_n + g_n \to f + g$ in measure. (You may assume either a general situation—all functions are defined on a measure space (S, Σ, μ), or that all functions are Lebesgue measurable on some $\mathbb{R}^d$ and the convergence is in Lebesgue measure. The proof will differ only in notation!

70. Tietze's extension theorem applied to $\mathbb{R}^d$ says: if f is a continuous function defined on a closed subset F of $\mathbb{R}^d$, then there exists a continuous function g defined on all of $\mathbb{R}^d$ such that g equals f on F. Let h be a Lebesgue measurable function defined on a Lebesgue measurable set E with $m_d(E) < \infty$. Show that for any ϵ there is a continuous function g_ϵ such that $m_d\left(\{x \in E\,;\, h(x) \neq g_\epsilon\}\right) < \epsilon$.

71. (Folland) Suppose $f_n \to f$ in measure and that $f_n \geq 0$ for all n. Conclude that $\int f\,d\mu \leq \liminf \int f_n\,d\mu$. (Hint: Use subsequences.)

72. (Folland) Suppose $|f_n| \leq g$, where $\int g\,d\mu < \infty$, and $f_n \to f$ in measure. Show that $\int f_n\,d\mu \to \int f\,d\mu$ and $f_n \to f$ in L^1.

73. If the sequence $\{f_n\}$ of positive, measurable functions is decreasing and converges in measure to f, then it converges a.e. to f.

74. Assume $\inf |f|\,d\mu < \infty$. Show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\mu(A) < \delta$, $\displaystyle\int_A |f|\,d\mu < \epsilon$. Hint: Note that $\displaystyle\lim_{K\to\infty} \int_{\{|f|>K\}} |f|\,d\mu = 0$. Why?

75. A sequence $\{f_n\}$ of functions on the measure space (S, Σ, μ) is said to be **uniformly integrable** if

- $\sup_n \int |f_n|\, d\mu < \infty$;

- For every $\epsilon > 0$, there is a K_ϵ such that $\sup_n \left\{ \int_{\{|f_n|>K_\epsilon\}} |f_n|\, d\mu \right\} < \infty$.

a) Show that if $|f_n| \leq g$ for every n, where $g \in L^1$, then $\{f_n\}$ is uniformly integrable.

b) For $\ell = 2^n + k$, where n is a nonnegative integer and $0 \leq k < 2^n$, let $f_\ell(x) = n^2 \chi_{[k/2^n,(k+1)/2^n]}(x)$ define a sequence of functions on $[0,1]$. Show that this sequence is uniformly integrable as a sequence of functions on $[0,1]$ with Lebesgue measure, but there is no L^1 function that dominates this sequence.

c) The sequence of functions $f_n(x) = n/(n^2 + x^2)$ is uniformly integrable on $\mathbb{R}$ with Lebesgue measure.

76. Show that if $\mu(S) < \infty$, $\{f_n\}$ is a uniformly integrable sequence of functions on (S, Σ, μ), and if $f_n \to f$ in μ measure, then $f_n \to f$ in L^1. (A study of L^1 convergence in example c) of the previous problem shows that the hypothesis $\mu(S) < \infty$ is needed.)

(Hint: a) Use an a.e. converging subsequence and Fatou's Lemma to conclude that $\int |f|\, d\mu < \infty$.

b) Show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\mu(A) < \delta$, $\int_A |f_n|\, d\mu < \epsilon$ for all n and also for f; see problem 74.

c) Prove the result partitioning S into the region $|f_n - f < \epsilon$ and its complement.)

77. Wheeden and Zygmund, page 96, problem 1(b).

78. Wheeden and Zygmund, page 96, problem 3.

79. Wheeden and Zygmund, page 96, problem 4.

80. Wheeden and Zygmund, page 97, problem 5.

81. Wheeden and Zygmund, page 97, problem 10.

82. (Folland). Let $f(x,y) = (1 - xy)^{-a}$, where $a > 0$. What can you say about existence and equality of the integrals $\int_{\mathbb{R}^2} f\, dm_2$, $\int_{[0,1]} \int_{[0,1]} f dx dy$, $\int_{[0,1]} \int_{[0,1]} f dx dy$.

83. (Folland)

a) By integrating $e^{-sxy} \sin x$ with respect to x and y, show that $\int_0^\infty e^{-sx} x^{-1} \sin x\, dx = \pi/2 - \arctan s$.

b) Show that $\int_{\mathbb{R}} |x^{-1} \sin x|\, dx = \infty$.

c) Show that $\int_0^\infty \triangleq \lim_{b\to\infty} \int_0^b x^{-1} \sin x\, dx = \pi/2$. Note: We cannot take the limit as $s \downarrow 0$ in the result of a) even thought this formally gives the correct result, since by b) the dominated convergence theorem is not available. Instead integrate with respect to x and y on $[0, b] \times [0, \infty)$ and pass to the limit.

84. Let g be a monotone, differentiable function mapping the interval I to $\mathbb{R}$. Prove that

$$\int_I f(g(y))|g'(y)|\, dm(y) = \int_{\mathbb{R}} f(x)\, dm(x).$$

85. Show that if f is either non-negative or integrable as a function on $\mathbb{R}^2$,

$$\int_{\mathbb{R}^2} f(z)\, dz = \int_0^\infty \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r\, d\theta\, dr.$$

You can do this using the tools we developed. By Fubini-Tonelli the integral over the upper half plane equals the iterated integral

$$\int_{[0,\infty]} \left[\int_{\mathbb{R}} f(x, y)\, dx \right] dy.$$

For each $y > 0$, use problem 84 to make the change of variable in the x integral: $\cot\theta = x/y$. Now use Fubini to interchange dy and $d\theta$ integration. Then for each s ($s \neq 0$, $s \neq \pi$, apply the change of variables. $r = y/\sin\theta$. Also address the measurability, in (r, θ) of $f(r\cos\theta, r\sin\theta)$.