501 Problems

45. Let A and B be Lebesgue measurable subsets of $\mathbb{R}$. Show that $A\times B$ is a Lebesgue measurable subset of $\mathbb{R}^2$.

46. Let $\mathcal{R}$ be an algebra of subsets of S. Let μ be a σ-finite measure on $(S, \sigma(\mathcal{R}))$. Show that for each $A \in \sigma(\mathcal{R})$ and $\epsilon > 0$ there is an A_ϵ in $\mathcal{R}$ with $\mu(A \triangle A_\epsilon) < \epsilon$.

47. (From Folland) Let $\mathring{m}$ be a finitely additive measure on $(S, \mathcal{R})$, where $\mathcal{R}$ is an algebra. Assume $\mathring{m}\,(S) < \infty$ and $\mathring{m}$ is continuous from below. Let μ^* be the outer measure induced by $\mathring{m}$. Define the inner measure of a set by $\mu_*(A) = \mu^*(S) - \mu^*(A^c)$. Show that A is μ^*-measurable if and only if $\mu^*(A) = \mu_*(A)$. See problem 44 for help.

48. Prove Theorem 3 in the Lecture Notes, *Construction of Measures*.

49. Let

$$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Let m_F^* be the outer measure induced by F—see Lecture Notes, *Lebesgue-Stieltjes Measures*. Determine the class m_F^* measurable sets and describe m_F^*.

50. Describe m_F and the collection of m_F^*-measurable sets in the following examples.

(a) $F(x) = \begin{cases} x, & \text{if } x < 0; \\ 1 + x, & \text{if } x \geq 0. \end{cases}$

(b) $F(x) = \begin{cases} 0, & \text{if } x < 0; \\ x, & \text{if } 0 \leq x < 2; \\ 2, & \text{if } x \geq 2; \end{cases}$

51. (Folland) **(a)** Suppose A is a Lebesgue measurable subset of $\mathbb{R}$ with positive Lebesgue measure. For any α such that $0 < \alpha < 1$, there is an open interval I with $m(A \cap I) > \alpha m(I)$.

(b) Let A be a Lebesgue measurable set with positive Lebesgue measure. Then $A - A$, which is defined to be the set $\{x - y\,;\, x \in A,\ y \in A\}$, contains an open interval. (Hint: If the interval I is as in part (a), show that $(-m(I)/2, m(I)/2)$ is contained in $A - A$.)

52. Let S be a metric space with metric d. An outer measure ν on S is called a **metric outer measure** if

$$\nu(A \cup B) = \nu(A) + \nu(B) \quad \text{whenever } \operatorname{dist}(A, B) > 0.$$

(Here $\operatorname{dist}(A, B) \stackrel{\triangle}{=} \inf\{d(x, y)\,;\, x \in A,\ y \in B\}$.)

Show that if F is right-continuous and increasing on $\mathbb{R}$, the outer measure m_F^* induced by F is a metric outer measure.

54. (Follow up to 53). This problem gives an alternative approach to proving that the Borel sets are m_F^*-measurable. The object is to prove:

Theorem 1 *If ν is a metric outer measure on a metric space S, then the Borel sets of S are ν-measurable. Conversely, if the Borel sets are ν-measurable, then ν is a metric outer measure.*

Fill in the details of the following proof.

(a) Prove that if the Borel sets are ν-measurable, then ν is a metric outer measure. Hint: If $\mathrm{dist}(A, B) > 0$, then $\bar{A} \cap B = \emptyset$, where $\bar{A}$ is closed. Use the ν-measurability of $\bar{A}$ to break up $\nu(A \cup B)$ into two parts.

Now assume that ν is a metric outer measure. To prove that the Borel sets are ν-measurable, it is enough to prove that the closed sets are Borel measurable. Thus let A be any closed set. It is necessary to show that

$$\nu(E) \geq \nu(E \cap A) + \nu(E \cap A^c) \quad \text{for every } E \subseteq S. \tag{1}$$

(Since the opposite inequality is automatically true by subadditivity of ν, one can then conclude equality holds.)

If $\nu(E) = \infty$, then (1) is true trivially. Hence, assume $\nu(E) < \infty$. Let $G_0 \triangleq \{x\,;\, \mathrm{dist}(x, A) > 1\} \cap E$. For $n \geq 1$, define

$$G_n = \left\{x\,;\, \frac{1}{n+1} < \mathrm{dist}(x, A) \leq \frac{1}{n}\right\} \cap E.$$

(b) Use the metric outer measure propery to prove

$$\sum_{k=0}^{\infty} \nu(G_{2k}) \leq \nu(E) < \infty, \qquad \sum_{k=0}^{\infty} \nu(G_{2k+1}) \leq \nu(E) < \infty.$$

(c) Use (b) and subadditivity to prove

$$\lim_{n\to\infty} \nu\left(E \cap \left\{x; \mathrm{dist}(x, A) > \frac{1}{n}\right\}\right) = 0.$$

Hint: $E \cap A^c = \left\{x; \mathrm{dist}(x, A) > \frac{1}{n}\right\} \cup [\cup_{m=n}^{\infty} G_m]$.

(d) For any n, $E \cap A$ and $E \cap \left\{x; \mathrm{dist}(x, A) > \frac{1}{n}\right\}$ are a positive distance apart and their union is contained in E. Complete the proof of (1).