1. Roberta plays a game in which her chance of winning is 40%. Using a normal approximation with half integer (continuity) correction, estimate, in explicit decimal form, the probability that if she plays the game 150 times she will win at most 55 times. (Note: the numbers in this problem work out to be simple.)

Solution: If X is the number of games that Roberta wins then X is binomial with $n = 150$ and $p = 0.4$, and so $E[X] = np = 60$, $\mathrm{Var}(X) = np(1-p) = 36$. $(X - 60)/6)$ is approximately standard normal, so

$$P(X \leq 55) = P(X \leq 55.5) = P\left(\frac{X-60}{6} \leq -0.75\right) = \Phi(-0.75) = 1 - \Phi(0.75) = 0.2266.$$

2. Two hundredths of one percent (0.0002) of the US population has a certain genetic defect. Use a Poisson approximation to estimate the probability that there is at most one person with that defect in a certain city with population 10000.

Solution: If X is the number of people in the city with the defect then X is binomial with $n = 10000$, $p = 0.0002$ and may be approximated by a Poiusson random variable Y with parameter $\lambda = np = 2$. Then

$$P(X \leq 1) \approx P(Y \leq 1) = P(Y = 0) + P(Y = 1) = \frac{2^0}{0!}e^{-2} + \frac{2^1}{1!}e^{-2} = 3e^{-2}.$$

3. A fair coin is flipped once. If a head is obtained, the experiment stops; if a tail is obtained, the coin is flipped two more times. Let X be the number of heads obtained on the first flip and Y the total number of heads obtained in the experiment.

(a) Via a table, or otherwse, give the joint probability mass function of X and Y.

(b) Find the (marginal) probability mass function p_Y for Y, and find $E[Y]$ and $\mathrm{Var}(Y)$.

(c) Find the probability mass function p_Z for the random variable $Z = X + Y$.

Solution: (a) The sample space can be taken to be a list of all experimental results:

$$S = \{H, THH, THT, TTH, TTT\},$$
$$P(H) = \frac{1}{2}, \qquad P(THH) = P(THT) = P(TTH) = P(TTT) = \frac{1}{8}.$$

This leads to the probabiity mass function $p(x, y)$ given in the table:

$x \backslash y$	0	1	2
0	1/8	1/4	1/8
1	0	1/2	0

(b) $P\{Y = 0\} = 1/8$, $P\{Y = 1\} = 3/4$, $P\{Y = 2\} = 1/8$, Then

$$E[Y] = 0\frac{1}{8} + 1\frac{3}{4} + 2\frac{1}{8} = 1, \qquad E[Y^2] = 0\frac{1}{8} + 1\frac{3}{4} + 4\frac{1}{8} = \frac{5}{4}, \quad \mathrm{Var}(Y) = \frac{5}{4} - 1^2 = \frac{1}{4}.$$

(c) The probabilities of various values of Z are obtained by adding the corresponding values in the table. Thus

$$P\{Z=0\} = p(0,0) = \frac{1}{8},$$
$$P\{Z=1\} = p(0,1) + p(1,0) = \frac{1}{4},$$
$$P\{Z=2\} = p(0,2) + p(1,1) = \frac{5}{8}.$$

4. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} C(4-x^2), & \text{if } 0 \le x \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate the normalizing constant C.

(b) Find $P\{X=1\}$ and $P\{X \ge 1\}$.

(c) Find $E[X]$ and $\mathrm{Var}(X)$.

Solution: (a) $1 = \int_{-\infty}^{\infty} f(x)\,dx = C\int_0^2 (4-x^2)\,dx = C[4x - x^3/3]_0^2 = 16C/3$, so $C = 3/16$.

(b) $P\{X=1\} = 0$, since X is a continuous random variable.

$P\{X \ge 1\} = (3/16)\int_1^2 (4-x^2)\,dx = (3/16)[4x - x^3/3]_1^2 = 5/16$.

(c)
$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{3}{16}\int_0^2 x(4-x^2)\,dx = \frac{3}{4};$$
$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{3}{16}\int_0^2 x^2(4-x^2)\,dx = \frac{4}{5};$$
$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{19}{80}.$$

5. An amateur astonomer observes an average of one shooting star per hour. In answering the following questions, assume that the occurrence of shooting stars is a Poisson process,

(a) What is the probability that he will see a shooting star in his first half hour of observation tonight?

(b) Tomorrow night he plans to carry out observations from 9:00 PM through 2:00 AM. What is the probabiity he will see the same number of stars before and after midnight? You may leave your answer as the sum of an infinite series.

Solution: (a) If X is the number of shooting stars seen in the first half hour tonight then X is a Poisson RV with parameter 1/2, so $P\{X \ge 1\} = 1 - P\{X=0\} = 1 - e^{-1/2}$.

(b) The numbers Y and Z of stars seen in beffore and after midnight are independent Poisson RV's with parameters 3 and 2, repectively. So

$$P\{Y=Z\} = \sum_{k=0}^{\infty} P\{Y=k\}P\{Z=k\} = \sum_{k=0}^{\infty} \frac{3^k}{k!}\,e^{-3}\,\frac{2^k}{k!}\,e^{-2} = \sum_{k=0}^{\infty} \frac{6^k}{(k!)^2}\,e^{-5}.$$

6. Suppose that X and Y are continuous random variables with joint density

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y), & \text{if } 0 \le x, y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the (marginal) density $f_X(x)$ of X; specify $f_X(x)$ for all real x.

(b) Find $P\{X \le Y\}$.

Solution: (a) Clearly $0 < X < 1$ and so $f_X(x) = 0$ if $x < 0$ or $x > 1$. If $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{2}{3}(x+2y)\,dy = \frac{2}{3}\Big[xy + y^2\Big]_0^1 = \frac{2(x+1)}{3}.$$

(b)
$$P\{X \le Y\} = \iint_{x \le y} f(x,y)\,dxdy = \frac{2}{3}\int_0^1 \int_0^y (x+2y)\,dx\,dy$$
$$= \frac{2}{3}\int_0^1 \left[\frac{x^2}{2} + 2xy\right]_0^y dy = \frac{2}{3}\int_0^1 \frac{5}{2}y^2\,dy = \frac{5}{9}.$$

7. Let X be an exponential random variable with parameter $\lambda = 2$, and let $Y = 2X + 5$.

(a) What are the possible values of Y?

(b) Find the cumulative distribution function $F_Y(y)$ of Y. Be sure to specify $F_Y(y)$ for all values of y.

(c) Find the probability density function f_Y of Y. Be sure to specify $f_Y(y)$ for all values of y.

Solution: (a) Since $X \ge 0$ we have $2X \ge 0$ and so $Y = 32X + 5 \ge 5$.

(b) From (a), $F_Y(y) = 0$ if $y < 5$. For $y \ge 5$,

$$F_Y(y) = P\{Y \le y\} = P\{2X + 5 \le y\} = P\{X \le \frac{y-5}{2}\}$$
$$= 2\int_0^{(y-5)/2} e^{-2x}\,dx = 1 - e^{-(y-5)}.$$

(c) From (a), $f_Y(y)$ is zero for $y < 5$. For $y \ge 5$,

$$f_Y(y) = F_Y'(y) = e^{-(y-5)}.$$