SUPPLEMENTARY QUESTIONS 3

The following exercise shows that the size of $\mathcal{P}(\mathbb{N})$ is not $\aleph_\omega$ (and more generally has uncountable cofinality.)

Question 1. Prove that it is impossible to express

$$\mathcal{P}(\mathbb{N}) = \bigcup_{n\in\mathbb{N}} S_n,$$

where each $S_n \prec \mathcal{P}(\mathbb{N})$.

Hint: In order to reach a contradiction, express

$$\mathbb{N} = \bigsqcup_{n\in\mathbb{N}} X_n$$

as a disjoint union of infinitely many infinite subsets. Notice that for each $n \in \mathbb{N}$,

$$P_n = \{A \cap X_n \mid A \in S_n\}$$

is a *proper* subset of $\mathcal{P}(X_n)$. Now continue with the appropriate diagonal argument!