(25) **1.** Solve the following LPP by the primal-dual method. No credit will be given for a solution by any other method. Indicate threshold values of μ on the axis at right. FOR EACH TABLEAU state explicitly the range of values of μ for which that tableau is feasible and dual-feasible (i.e. represents an optimal solution).

$$\text{Maximize } z = x_1 + 2x_2 \text{ subject to}$$
$$x_1 - 2x_2 \geq 3$$
$$3x_1 - 2x_2 \leq 10$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution: The tableaux are given below. The first tableau is FDF for $\mu \geq 3$. The first threshold is $\mu = 3$. Dual simplex pivot on -1.
The second tableau is FDF for $4/3 \leq \mu \leq 3$. The second threshold is $\mu = 4/3$. Primal simplex pivot on 4.
The third tableau is FDF for $0 \leq \mu \leq 4/3$. Set $\mu = 0$ to obtain the final tableau for the given LPP.

	x_1	x_2	u_1	u_2	
u_1	-1	2	1	0	$-3+\mu$
u_2	3	-2	0	1	$10+\mu$
	$-1+\mu$	$-2+\mu$	0	0	

$\longrightarrow$

	x_1	x_2	u_1	u_2	
x_1	1	-2	-1	0	$3-\mu$
u_2	0	4	3	1	$1+4\mu$
	0	$-4+3\mu$	$-1+\mu$	0	

$\longrightarrow$

	x_1	x_2	u_1	u_2	
x_1	1	0	0.5	0.5	$3.5+\mu$
x_2	0	1	0.75	0.25	$0.25+\mu$
	0	0	$2-1.25\mu$	$1-0.75\mu$	

$\longrightarrow$

	x_1	x_2	u_1	u_2	
x_1	1	0	0.5	0.5	3.5
x_2	0	1	0.75	0.25	0.25
	0	0	2	1	4

The optimal solution is $x_1 = 3.5$, $x_2 = 0.25$, $z = x_1 + 2x_2 = 4$.

(15) **2.** Use a cutting plane method to solve the following pure integer programming problem:

$$\text{Maximize } z = 100x_1 + 300x_2 \text{ subject to}$$
$$-2x_1 + 4x_2 \leq 12$$
$$3x_1 - x_2 \leq 10$$
$$x_1,\ x_2 \geq 0$$
$$x_1,\ x_2 \text{ integers}$$

To start you off, here is the final tableau for the solution of the relaxed LPP:

	x_1	x_2	u_1	u_2	
x_1	1	0	0.1	0.4	5.2
x_2	0	1	0.3	0.2	5.6
	0	0	100	100	2200

Solution: Since x_1 and x_2 are both nonintegral, both rows are candidates to produce the cutting plane. Using the largest fractional part rule, we choose the second constraint (5.6) to generate the cutting plane. The pure integer Gomory cutting plane formula gives

	x_1	x_2	u_1	u_2	u_3	
x_1	1	0	0.1	0.4	0	5.2
x_2	0	1	0.3	0.2	0	5.6
u_3	0	0	−0.3	−0.2	1	−0.6
	0	0	100	100	0	2200

$\longrightarrow$

	x_1	x_2	u_1	u_2	u_3	
x_1	1	0	0	1/3	1/3	5
x_2	0	1	0	0	1	5
u_3	0	0	1	2/3	−10/3	2
	0	0	0	100/3	1000/3	2000

The dual simplex method has led to the optimal solution $x_1 = 5$, $x_2 = 5$, $z = 2000$.

(15) **3.** Consider the following integer programming problem, which is the same as in the previous problem EXCEPT THAT the integrality condition only requires x_1 to be an integer.

$$\text{Maximize } z = 100x_1 + 300x_2 \text{ subject to}$$
$$-2x_1 + 4x_2 \le 12$$
$$3x_1 - x_2 \le 10$$
$$x_1,\ x_2 \ge 0$$
$$x_1 \text{ integer}$$

Solve this mixed IPP by "branch and bound". Show the decision tree, labelled fully and clearly, as well as any tableaux you may develop. Use the same starting point as the previous problem and explain why this is appropriate, even though the problems are different.

Solution: The root of the tree is the optimal solution of the relaxed LPP: $x_1 = 5.2$, $x_2 = 5.6$, $z = 2200$. Initially this root is dangling since x_1 does not satisfy the integrality requirement. There is no choice about how to branch: the only problem variable is x_1, so create two new problems by adding constraints $(A):\ x_2 \le 5$ and $(B):\ x_2 \ge 6$. The new constraint requires a new slack or surplus variable u_3.

For (A) we get the row 1 0 0 0 1 5, and subtracting the top row of the given tableau we get

	x_1	x_2	u_1	u_2	u_3	
x_1	1	0	0.1	0.4	0	5.2
x_2	0	1	0.3	0.2	0	5.6
u_3	0	0	−0.1	−0.4	1	−0.2
	0	0	100	100	0	2200

$\longrightarrow$

	x_1	x_2	u_1	u_2	u_3	
x_1	1	0	0	0	1	5
x_2	0	1	0.25	0	0.5	5.5
u_3	0	0	0.25	1	−2.5	0.5
	0	0	125	0	250	2150

So the (A) branch is $x_1 = 5$, $x_2 = 5.5$, $z = 2150$. This satisfies the integrality condition.
For the (B) branch the row is $-1\,0\,0\,0\,1 - 6$, and adding the top row of the given tableau gives

	x_1	x_2	u_1	u_2	u_3	
x_1	1	0	0.1	0.4	0	5.2
x_2	0	1	0.3	0.2	0	5.6
u_3	0	0	0.1	0.4	1	-0.8
	0	0	100	100	0	2200

The u_3 row shows that problem (B) is infeasible.
Now (A) is Best So Far, while (B) is infeasible and hence Out Of Contention. Therefore, the optimal solution is $x_1 = 5$, $x_2 = 5.5$, $z = 2150$.

(15) **4.** In this question you should use DUALITY THEORY to get your answers. No credit will be given for answers based on simplex, dual-simplex, or primal-dual methods, or for any computations or observations based on tableaux.
The following LPP is called (P):

$$\begin{aligned} \text{Maximize } 25x_1 + 3x_2 + 40x_3 \text{ subject to} \\ 2x_1 - x_2 + 4x_3 &\le 14 \\ 4x_1 + 3x_2 + 6x_3 &\le 22 \\ x_1,\ x_2,\ x_3 &\ge 0 \end{aligned}$$

(a) Formulate the dual problem (D).
Solution:

$$\begin{aligned} \text{Maximize } 14w_1 + 22w_2 \text{ subject to} \\ 2w_1 + 4w_2 &\ge 25 \\ -w_1 + 3w_2 &\ge 3 \\ 4w_1 + 6w_2 &\ge 40 \\ w_1,\ w_2,\ w_3 &\ge 0 \end{aligned}$$

(b) Suppose that you have the following information about an optimal solution $\boldsymbol{x}$ for (P) and an optimal solution $\boldsymbol{w}$ for (D):

$$x_1 = 1 \text{ and } w_2 = 5.$$

This information guarantees that certain constraints in (P) and (D) have no slack at $\boldsymbol{x}$ and $\boldsymbol{w}$, respectively. Which constraints are these? Explain your reasoning (very briefly!).
Solution: By complementary slackness, the first slack variable in (D) and the second slack variable in (P) are 0. So $4x_1 + 3x_2 + 6x_3 = 22$ and $2w_1 + 4w_2 = 25$.
(c) Use complementary slackness to find the values of as many of the variables w_1, w_2, z_D at the optimal solution of (D).
Solution: $\boldsymbol{w} = [2.5\ 5]^T$ and $z_D = 145$. Reasoning: By the previous part, $2w_1 + 4w_2 = 25$. Given $w_2 = 5$, we get $w_1 = 2.5$ and then $z_D = 14w_1 + 22w_2 = 35 + 110 = 145$.

(d) Do the same for x_1, x_2, x_3, z_P at the optimal solution of (P).
Solution: $\boldsymbol{x} = [1\,0\,3]^T$ and $z_P = 145$. Reasoning: We are given $x_1 = 1$. From part (c), there is nonzero slack in the second dual constraint, because $-w_1 + 3w_2 = -2.5 + 3 \cdot 5 = 12.5 > 3$. By complementary slackness, $x_2 = 0$. By part (b), $4x_1 + 3x_2 + 6x_3 = 22$. Therefore $4 \cdot 1 + 0 + 6x_3 = 22$ so $x_3 = 3$. Finally $z_P = z_D = 145$ by the Duality Theorem.

(20) **6.** A certain LPP in standard form leads to the following final tableau:

	x_1	x_2	x_3	u_1	u_2	
	1	4.5	0	−1.5	1	2
	0	−2.5	1	1	−0.5	6
	0	19	0	5	10	580

The objective function is $z = 50x_1 + 6x_2 + 80x_3$.
(a) Imagine changing the LPP by changing the first coefficient c_1 of the objective function, but changing nothing else. For which values of c_1 will the basic solution $x_1 = 2$, $x_2 = 0$, $x_3 = 6$ still be an optimal solution? Your answer should be a range of values of c_1.
Solution: Since x_1 is basic, the answer depends on the dual θ-ratios with respect to nonbasic entries in the x_1-row. These are

$$\frac{19}{4.5}, \ \frac{5}{-1.5}, \ \frac{10}{1}.$$

Changing signs and choosing the values closest to 0 on each side gives the condition

$$\frac{-19}{4.5} \le \Delta c_1 \le \frac{5}{1.5}, \quad \text{that is,} \quad 50 - \frac{19}{4.5} \le c_1 \le 50 + \frac{5}{1.5}, \quad \text{or} \quad \frac{412}{9} \le c_1 \le \frac{480}{9}.$$

(b) If the original resource column is changed to $[20\ 60]^T$, will the optimal solution of the new problem still have x_1 and x_3 as basic variables? Explain your reasoning.
Solution: It was intended that u_1 and u_2 be slack variables, though that should have been explicitly stated. With that assumption,

$$B^{-1} = \begin{bmatrix} -1.5 & 1 \\ 1 & -0.5 \end{bmatrix}, \quad \text{and the new resource column } = B^{-1}\begin{bmatrix} 20 \\ 60 \end{bmatrix} = \begin{bmatrix} 30 \\ -10 \end{bmatrix}.$$

Because of the negative entry, the tableau of the new problem in which x_1 and x_3 are basic is an infeasible tableau. Therefore, the answer is "no."
(c) For the dual problem of the original LPP, what is the value of the second dual variable w_2 at the optimal solution?
Solution: $w_2 = 10$. This is because w_2 corresponds to the primal variable u_2 (which could be called w_2^*). By the correspondence between primal and dual tableaux, w_2 is the entry in the primal objective row for u_2.