We begin by restating an earlier problem.

4*. [Old] Let $S = \{0, 1, 2, 3\}$, and model $S_4$ by $A(S)$. Let $i$ be the identity, given by $in = n$ for $n \in S$; let $f$ be defined by

$$f0 = 1, \quad f1 = 2, \quad f2 = 3, \quad f3 = 0;$$

and let $g$ be defined by

$$g0 = 1, \quad g1 = 0, \quad g2 = 2, \quad g3 = 3.$$

Find the powers of $f$, and show that $f^4 = i$. [New] Let

$$H = \{ h \in A(S) : h1 = 1 \}.$$

(a) Show that $H$ is a subgroup. Elements of $H$ can be viewed as permutations of $\{1, 2, 3\}$, and all such permutations arise, so $H$ can be considered as $S_3$. Thus $H$ has order 6.

(b) The proof of Lagrange’s Theorem shows that there are four left cosets $gH$. To aid in characterizing these cosets, show that, if $g_0$ and $g_1$ lie in the same coset, then $g_0 0 = g_1 0$.

(c) Show that $f^0 = i$, $f^1 = f$, $f^2$ and $f^3 = f^{-1}$ lie in different cosets. Since we now have four representatives of four cosets, every element of $A(S)$ lies in one of $iH$, $fH$, $f^2H$, $f^{-1}H$.

(d) Show that $g \in fH$, so that $f^{-1}g \in H$. Also identify the cosets containing each $gf^i$, $i = 0, 1, 2, 3$ Using the appropriate coset representative, this gives elements of the form $f^jgf^i \in H$.

(e) Our previous study of $S_3$ (p. 17 of textbook), when applied to $H$ shows that we could obtain all elements of $H$ from two of those found in (d). From this, conclude [old] that all 24 elements of $A(S)$ can be obtained as products of terms, each of which is $f$ or $g$. (Now, you shouldn’t need a hint.)

5. Suppose that $z_0$ and $z_1$ are complex numbers and $|z_0| = |z_1| = 1$.

(a) Show that $|z_0 + z_1| \leq 2$.

(b) If $\alpha$ is a complex number with $|\alpha| \leq 2$, show how to find complex numbers $z_0$ and $z_1$ with $|z_0| = |z_1| = 1$ and $z_0 + z_1 = \alpha$.

(c) Since $|z_0| = |z|$, there are infinitely many pairs $(z_0, z_1)$ satisfying (b) if $\alpha = 0$; how many solutions are there if $\alpha \neq 0$?

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6. (Based on problems in Section 2.2 of textbook). For some fixed integer $n$, suppose that $G$ is a group such that, for all $a$ and $b$ in $G$, $(ab)^n = a^n b^n$. The identity element of $G$ will be denoted $e$, all other letters are available to use as variables — and we will take advantage of that (it is more customary to overwork a few letters, but arguments are easier to follow if the general variables used to express a formula are never used to name individuals to which the formula applies).

(a) Show that, for all $c$ and $d$ in $G$,

$$(cd)^{n-1} = d^{n-1} c^{n-1}$$

(b) Show that, for all $f$ and $g$ in $G$,

$$f^n g^{n-1} = g^{n-1} f^n$$

(c) Show that, for all $h$ and $k$ in $G$,

$$(hkh^{-1}k^{-1})^n = hkh^{-1}k^{-n}$$

$$(hkh^{-1}k^{-1})^n = h^n kh^{-n} k^{-1}$$

$$(hkh^{-1}k^{-1})^{n-1} = k^{1-n} h^{n-1} k^{-n-1}$$

$$(hkh^{-1}k^{-1})^{n-1} = k^{1-n} h^{n-1} k^{-1}$$

(See hint for problem 6c on page 50 of the textbook).

(d) If $p$ and $q$ are in $G$, find many expressions for

$$(pq)^{n-1} p^{-1} q^{-1}$$

(e) Show that, for all $p$ and $q$ in $G$,

$$(pq)^{n(n-1)} p^{-1} q^{-1} = e$$

End workshop 2