1. Suppose that the group $G$ acts on the set $S$, and $\alpha \in S$. The set $\text{Orb}(\alpha) := \{g \alpha \mid g \in G\}$ is called the orbit of $\alpha$ under $G$, or just the orbit of $\alpha$. An orbit of $G$ on $S$ is defined to be a subset of $S$ of the form $\text{Orb}(\beta)$ for some $\beta \in G$.
   (a) Show that for any $\alpha, \beta \in S$, either $\text{Orb}(\alpha) = \text{Orb}(\beta)$ or $\text{Orb}(\alpha) \cap \text{Orb}(\beta) = \emptyset$.
   (b) Show that $G$ acts transitively on $\text{Orb}(\alpha)$.
   (c) Show that for all $g, h \in G$, 
   \[ g \alpha = h \alpha \iff g \equiv h \pmod{G_\alpha}. \]
   (d) (THE “COUNTING PRINCIPLE”) Conclude that if $G$ and $S$ are finite, then $|\text{Orb}(\alpha)| = |G|$.
   (Reality check: what does this assertion mean in the case that $g \alpha = \alpha$ for all $g \in G$?)
   (e) Show that if $\alpha_1, \alpha_2, \ldots, \alpha_m \in S$ and every orbit of $G$ on $S$ contains $\alpha_i$ for exactly one value of $i$, then 
   \[ |S| = \sum_{i=1}^{m} |G|/|G_{\alpha_i}|. \]

2. This problem involves groups, orbits, fixed points, patterns, & jewelry. Suppose that you are a junk jewelry manufacturer. You make bracelets by attaching fake stones at intervals of $2\pi/5$ radians around a wire circle. The stones are all identical, except that they come in $m$ colors: red, purple, clear, green, blue, ... Two bracelets are considered to have the same pattern if you can lay them next to each other on a table so that the pattern of colors is identical. What is the number $N(m)$ of different patterns?

   With one color only, clearly $N(1) = 1$. To make sure you understand the situation, count $N(2)$ by hand. You should get $N(2) = 8$, but even in this simple case it is easy to make a mistake. If by brute force you manage to get the correct number $N(3) = 39$ for 3 colors, then you are an excellent counter!

   There is a better way.

   This is a question about orbits. There are $m^5$ ways to choose the colors, but some of these give bracelets with the same pattern. In fact two bracelets have the same pattern if and only if they become identical when “set down on a table correctly”. This means that one can be transformed to the other by applying a symmetry of a regular pentagon. Put a different way, the group $G = D_5$ acts on the set $S$ of $m^5$ bracelets with all possible color choices, and the number of different possible patterns is the number of orbits of $G$ on $S$.

   The sweetest way to count orbits is based on “Burnside’s Lemma”, later refined to “Pólya’s Enumeration Theorem.” Develop Burnside’s Lemma as follows. Suppose that the finite group $G$ acts on the finite set $S$. Define 
   \[ \Theta(g) = |\{ \alpha \in S \mid g \alpha = \alpha \}| \text{the number of points in } S \text{ that are fixed by } g. \]
   (a) Let $P = \{(g, \alpha) \in G \times S \mid g \alpha = \alpha\}$. You will “count $|P|$ two ways”. First, show that $|P| = \sum_{g \in G} \Theta(g)$.
   (b) Use the definition of $G_\alpha$ from the previous problem. Show that if $\alpha \in S$, $g \in G$, and $\beta = g \alpha$, then $G_\beta = gG_\alpha g^{-1}$ (which is $\{g^{-1}xg \mid x \in G_\alpha\}$ by definition) and $|G_\beta| = |G_\alpha|$.
   (c) Suppose that $G$ acts transitively on $S$. Show that $|P| = |G|$, counting $|P|$ in a different way from (a). Conclude that 
   \[ \frac{1}{|G|} \sum_{g \in G} \Theta(g) = 1. \]
   (d) Suppose that $G$ acts (but not necessarily transitively) on $S$. Generalize your calculation from (c) to show:
   \[ \text{number of orbits of } G \text{ on } S = \frac{1}{|G|} \sum_{g \in G} \Theta(g). \]

   This equation is Burnside’s Lemma.
   (e) List the elements of $D_5$ and for each $g \in D_5$, calculate the number $\Theta(g)$ of the $m^5$ color combinations that are fixed by $g$. For example, $\Theta(e) = m^5$.
   (f) Answer the original question: $N(m) = ????