1 List all the Euclidean domains that you know, specifying the \( \delta \) for each.

2 For which of the following properties is it true that if a ring \( R \) has the property, then the polynomial ring \( R[X] \) has that property?
   (a) \( R \) is commutative
   (b) \( R \) is not commutative
   (c) \( R \) is an integral domain
   (d) \( R \) is a field
   (e) \( R \) is a division ring
   (f) \( R \) is a PID
   (g) \( R \) is a Euclidean domain.

3 For which of the above properties is it true that if \( R \) and \( S \) are rings having the property, then \( R \times S \) has the property?

4 For which of the following properties is it true that if a ring \( R \) has the property and \( I \) is any ideal of \( R \) such that \( I \neq R \), then the quotient ring \( R/I \) has the property?
   (a) \( R \) is commutative
   (b) \( R \) is an integral domain
   (c) \( R \) is a field
   (d) Every ideal in \( R \) is principal.

5 Let \( R = \mathbb{Z}[X] \). Show that \( R/3R \cong \mathbb{Z}_3[X] \).

6 Let \( R = \mathbb{Z}_5[X] \) and \( I = (X^2) \). Let \( S = R/I \).
   (a) How many elements are there in \( S \)?
   (b) How many units are there in \( S \)?
   (c) How many zero-divisors are there in \( S \)?
   (d) True or false: every zero-divisor in \( S \) is nilpotent.
   (e) How many ideals are there in \( S \)?

7 Prove that \( \mathbb{Z}[i] \cong \mathbb{Z}[X]/(X^2 + 1) \).

8 Suppose that \( \phi : R \to S \) is a homomorphism of rings. Let \( I \) be any ideal of \( R \) such that \( I \subseteq \ker(\phi) \). Construct a ring homomorphism \( \overline{\phi} : R/I \to S \) such that \( \operatorname{im}(\overline{\phi}) = \operatorname{im}(\phi) \).

9 What is the result \( a = qb + r \) when the division algorithm in \( \mathbb{Z}[i] \) is applied with \( a = 12 + 8i \) and \( b = 4 + 7i \)? (Hint. Check the proof that \( \mathbb{Z}[i] \) is Euclidean.)

10 Add the appropriate hypothesis and prove the resulting statement.
   If \( R \) is a ring, ________________, and if \( a, b \) are nonzero elements of \( R \) such that \( a \mid b \) and \( b \mid a \), then there is \( u \in U(R) \) such that \( b = ua \).

11 Show that if \( R \) is a ring and \( \phi : R \to R \) is a ring homomorphism, then \( S := \{ r \in R \mid \phi(r) = r \} \) is a subring of \( R \). If \( R \) is a field, then is \( S \) necessarily a field?

12 Suppose that \( R \) is a PID, and let \( a, b \in R \).
(a) Does there necessarily exist a gcd $d$ of $a$ and $b$ in $R$? (What does this terminology mean, by definition?)
(b) If $d$ exists, in what sense (if any) is it unique?
(c) If $d$ exists, can $d$ be expressed as an $R$-linear combination of $a$ and $b$?
(d) Suppose that $S = R[X]$, or more generally suppose that $R$ is a subring of another integral domain $S$ (not necessarily a PID). Show that $d$ is a gcd of $a$ and $b$ in $S$. Your answer to (c) should be used somewhere.

13 Let $D_5$ be the symmetry group of a regular pentagon. How many cyclic subgroups does $D_5$ have? How many noncyclic subgroups?

14 True or false: If $G$ is a group, then the following cancellation law holds: for any $g, h, x \in G$, $gx = hx$ implies $g = h$.

15 Let $G$ be a group. Show that $G$ is abelian if and only if the mapping $\phi : G \to G$ defined by $\phi(g) = g^2$ is a homomorphism.

16 Let $G$ be a group and $x, y \in G$. Show that if $xy = yx$, then $xy^{-1} = y^{-1}x$.

17 Proof or counterexample: If $G$ is a group, $x, y \in G$, $n \in \mathbb{N}$, and $x$ and $y$ have order $n$, then $(xy)^n = 1$.

18 Suppose that $G$ is a finite group, and $H$ and $K$ are distinct subgroups of $G$ such that $|H| = |K|$. Show that $H \cup K$ is not a subgroup of $G$, but $H \cap K$ is a subgroup of $G$. 