1. $\sqrt{-1}$?
   
   (a) Find the order of all the elements of $U_{13}$. Try not to make 12 different calculations.
   
   Hint: if $a$ has order $n$, then $a^k$ has order $n/(k,n)$. So whenever you find the order of one element you should know the order of a whole bunch.
   
   (b) Find a generator of $U_{29}$, and use it to solve the equation $x^2 = -1$ in $Z_{29}$.
   
   Hint: $-1$ has order 2. So it would be the 14th power of an element of order 14.
   
   This is not an efficient way to solve the equation, but the idea is important.

2. We’ll need this one in a month
   
   Find all the subgroups of $U_{17}$; list them explicitly, from the largest to the smallest.
   
   Here’s a plan for you:
   
   (a) Find a generator of the group (order 16).
   
   (b) Find all the subgroups of $Z_{16}$ (another cyclic group of order 16, easier to work with).
   
   (c) Find a recipe for converting the answers to part (b) into answers to the original question.
   
   We will need this information when we study the construction of the regular 17-gon by ruler and compass.

3. $\sqrt{1}$!
   
   Show that any finite group of even order contains an element of order 2.
   
   (Hint: Study the equation $x^2 = e$. We know one solution, namely $x = e$, and we have to look for another one.
   
   Rewrite the equation as $x = x^{-1}$ and try to show first that there are an even number of elements in the group which do not (!!) satisfy this equation.
   
   Remark: actually for any prime $p$ which divides the order of a finite group $G$, there is an element of $G$ of order $p$. This is Cauchy’s theorem.

4. Some matrix groups:
   
   (a) Make up a full multiplication table for $SL(2, Z_2)$.
   
   (b) Show that $SL(2, 2Z)$ is not a group.
   
   (In fact, you can show that $SL(2, 2Z)$ doesn’t contain any elements at all.)
   
   (c) Let $R$ be a commutative ring with identity. Show that $SL(2, R)$ is a group.
   
   (You are going to have to assume that the formula $\text{det}(AB) = \text{det}(A) \text{det}(B)$ works for an arbitrary commutative ring, and also that the usual formula for inverting a matrix $A$ works whenever $\text{det}(A)$ is a unit. These points are generalizations of familiar facts from Linear Algebra, and we will not discuss them.)
   
   (d) Compute the order of $GL(2, Z_p)$ for $p$ a prime.
   
   (Use linear algebra over $Z_p$. What you actually need to “count” are bases for a 2-dimensional space with coefficients in $Z_p$.)
   
   (e) Let $R$ be an infinite commutative ring with identity. Prove that $SL(2, R)$ is an infinite group.

5. Where did you lose them?
   
   Find all the subgroups of $S_3$ (not hard) $S_4$ (laborious)
   
   Helpful theoretical information:
   
   (A) By Lagrange’s theorem, the order of a subgroup divides the order of the whole group.
   
   (B) Any group of prime order is cyclic.