The construction of a regular pentagon

We treat only the algebraic part of the problem, not the question of finding a good geometrical construction. Euclid solves the problem geometrically in Book IV of The Elements of Geometry (written in Alexandria, Egypt, in Greek, about 300 B.C., based on material that had probably been known for several centuries before that).

Algebraically, we need to find a solution in \( \mathbb{C} \) to the conditions
\[
z^5 = 1; \quad z \neq 1
\]
This is a root of the cyclotomic polynomial
\[
\frac{z^5 - 1}{z - 1} = z^4 + z^3 + z^2 + z + 1.
\]
Recall that this polynomial is irreducible, so that the degree \([\mathbb{Q}[z] : \mathbb{Q}] = 4\) for any root of this polynomial. Since 4 is a power of 2, we cannot say definitely whether this number is constructible, but we may suspect that it is.

(Actually, most irreducible equations of degree 4 do not have constructible roots, but this one is very special.)

First, an exercise about quadratic equations.

Exercise 1
Suppose \( r + s = a \) and \( rs = b \). Then \( r \) and \( s \) are the roots of the quadratic equation \( x^2 - ax + b = 0 \).

Now, here is Gauss’ method for solving the equation using square roots.

(1) Work out the subgroups of \( U_5 \), and place them one inside another:
\[
\{1\}; \{1, 4\}; \{1, 2, 3, 4\}
\]

(2) Arrange the elements of \( U_5 \) so that the subgroups can be seen, going from left to right:
\[
[1|4|2, 3]
\]

Consider the following numbers associated with these groups:
\( a = z + z^4 + z^2 + z^4, \)
\( b_1 = z + z^4, \)
\( b_2 = z^2 + z^3, \)
\( c_{11} = z, \)
\( c_{12} = z^4, \)
\( c_{21} = z^2, \)
\( c_{22} = z^3. \)

Let’s look at what we have done. In each case, we sum certain powers of \( z \), and there is a rule for which powers we take.

- **a**: This corresponds to the subgroup \( \{1, 4, 2, 3\} = U_5 \) and we take all powers, accordingly.
- **b**: Here \( b_1 \) corresponds to the subgroup \( \{1, 4\} \) and \( b_2 \) corresponds to the “other” elements \( \{2, 3\} \).
- **c**: Here \( c_{11} \) corresponds to the tiny subgroup \( \{1\} \) so \( c_{11} \) is just \( z \). \( c_{12} \) is the “other” element of \( \{1, 4\} \).

Similarly, \( c_{22}, c_{23} \) breaks up \( b_2 \) the way \( c_{11}, c_{12} \) breaks up \( b_1 \).

What is so special about these numbers?

**First**, \( a \) is rational, in fact \( a = -1 \).

**Second**, \( b_1 \) and \( b_2 \) are roots of a single quadratic equations with rational coefficients. To see this, use exercise 1: it is enough to check that \( b_1 + b_2 \) and \( b_1 \cdot b_2 \) are rational.

\( b_1 + b_2 = a \) (this is the point), and we know that is rational.

\( b_1 \cdot b_2 = (z + z^4) \cdot (z^2 + z^3) = z^3 + z^4 + z + z^2 = a \) as well (slightly miraculous, and we need to analyze this).

**Third**, \( c_{11} \) and \( c_{12} \) satisfy a quadratic equation with coefficients in \( \mathbb{Q}[b_1, b_2] \), and so do \( c_{21} \) and \( c_{22} \).

Let’s check the sums and products.
Sums: $c_{11} + c_{12} = b_1$, $c_{21} + c_{22} = b_2$. That is easy enough, as we set it up that way. Products $c_{11} \cdot c_{12} = 1$, $c_{21} \cdot c_{22} = 1$, slightly miraculous again.

Anyway, since $c_{11} = z$, we know now that $z$ is constructible from $b_1$ and $b_2$, and $b_1, b_2$ are constructible, so $z$ is constructible.

Let’s work out the formulas.

$b_1 + b_2 = -1$, $b_1 b_2 = -1$, so these are roots of the quadratic equation

$$x^2 + x - 1 = 0$$

that is

$$\frac{-1 \pm \sqrt{5}}{2}$$

Secondly $c_{11} + c_{12} = b_1$, $c_{11} \cdot c_{12} = 1$, so these are roots of the quadratic equation

$$x^2 - b_1 x + 1 = 0$$

that is

$$\frac{b_1 \pm \sqrt{b_1^2 - 4}}{2}$$

All together we get 4 possible roots, which I calculate as follows (this should be checked):

$$\frac{-1 \pm \sqrt{5} \pm i\sqrt{10 \pm 2\sqrt{5}}}{4}$$

In particular one finds

$$\cos(72^\circ) = \frac{-1 + \sqrt{5}}{4}; \quad \sin(72^\circ) = \sqrt{\frac{10 + 2\sqrt{5}}{4}}$$

assuming we have calculated properly up to this point.