Mathematics 251: Lab 4   TRIPLE INTEGRALS

Please turn in only the printout of your Maple worksheet, which should include the Maple commands you input and Maple's response. Use the **text** feature of Maple to add a header containing your name and explicit answers to all questions asked (do NOT write any of this material in by hand). Use the **title** option in all plots to introduce a label that will be kept with the plot when your report is printed.

The worksheet in the *seed file* is divided into **Sections** corresponding to the parts of this project description. It also contains almost all you need to get started; and imitating those instructions should allow you to complete the other parts. You may elaborate on this organization in preparing your report. Also, remove from the worksheet any extraneous material and any errors you have made.

In this lab, we use Maple to help visualize and compute the value of a triple integral. In particular, we see that choosing the order of integration carefully can simplify the process.

The basic command for integration in Maple is `int`. To evaluate $\int_0^1 (x+1)\,dx$, we type
`int(x+1, x=0..1);`.
To evaluate the double integral $\int_0^1 \int_0^{1-x} (xy + x + 1)\,dy\,dx$, we type
`int(int(x*y + x +1, y=0..1-x),x=0..1);`.
To evaluate the triple integral $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (xyz + x + 1)\,dz\,dy\,dx$, we type
`int(int(int(x*y*z + x + 1, z=0..1-x-y),y=0..1-x),x=0..1);`

1. If a solid has mass density $\delta = 1$, then its moment of inertia about the z-axis is given by

$$I_z = \int\int\int (x^2 + y^2)\,dV.$$

We wish to calculate I_z for a solid S defined to be the region between

$$y = 8 - z^2 - 2x^2 \qquad (1) \qquad \text{a paraboloid}$$

and

$$y = z^2 \qquad (2) \qquad \text{a parabolic cylinder}$$

1a. To see what these surfaces looks like, and how they intersect, plot the two surfaces given above on the same set of axes for $-2 \le x \le 2$, $-\sqrt{8} \le z \le \sqrt{8}$. The basic commands for defining this rectangle and plotting the surfaces are

```
y1:=8-z^2-2*x^2; y2:=z^2;
Rect:=x=-2..2,z=-sqrt(8)..sqrt(8);
plot3d({y1,y2},Rect);
```

and are given in the seed file, but **you should add a title** to the `plot3d` instruction.

1b. Since the equation of each boundary of S is given by an equation expressing y as a function of x and z, it is easiest to calculate I_z by doing the y integration first. What is the upper limit of the y integral? What is the lower limit? (Answer in **text**.)

1c. We also need to determine the x and z limits of integration. To get these, it may be helpful to view the projection of the intersection of the paraboloid and the parabolic cylinder onto the x-z plane. To get this projection, use the `implicitplot` command:

```
implicitplot(y1=y2,Rect,scaling=CONSTRAINED);
```

What type of curve is this projection of the intersection of the two surfaces? Is the intersection itself the same type of curve? (Answer in **text**.)

1d. Find bounds on z depending on x and constant bounds on x that describe this region that will be used in computing the integral. Test your bounds by repeating the `plot3d` command of 1a with `Rect` replaced by this exact description of the projection in the xz plane. The result should be resemble a solid body (limitations of computation may leave a small hole in the surface).

1e. Now formulate and evaluate the triple integral using the Maple command `int`.

Note: If you were doing this by hand, you would probably choose another way to compute the integral with respect to x and z, but it is better to use the most direct description when Maple is doing the computation.

2. Suppose now we want to compute the same integral by doing the z integration first. It is then useful to visualize the solid with x and y as independent variables. One way to do this would be to create another version of the plot in part 1d using solid colors for the graphs of `y1` and `y2` and `BOXED` axes. This figure can be moved so that you view it along the z axis to see the projection in the xy-plane and the curve separating the graphs of `y1` and `y2`. A different approach is taken in this problem. Your report should follow the outline below, but an alternate approach in a worksheet that you don't submit may be used to check your work.

2a. To do the z integration first, solve (by hand computation) equations (1) and (2) for z, assigning the names `z1` and `z2` to the Maple expressions for the positive and negative solutions, respectively, of equation (1) and the names `z3` and `z4` to the Maple expressions for the positive and negative solutions, respectively, of equation (2). These four equations each specify a portion of the boundary of S. Plot these four surfaces on a single set of axes over the region $-2 \le x \le 2$, $0 \le y \le 8$.

It may be useful to view the plot of part (2a) from several viewpoints to do the remaining parts of this assignment.

2b. The difficult part is now to determine the limits of integration in x and y. The projection of S into the xy-plane. The set S consists of all points (x, y) for which **all four functions** graphed in (2a) are defined. You should see that this is a region between the line $y = 0$ and a curve that has an equation of the form $y = f(x)$. Determine (by hand) $f(x)$ and assign the name `f` to the Maple expression for $f(x)$.

2c. Note that the plot in (2a) shows that for some values of x and y the z values inside S range from the bottom of the paraboloid to the top of the paraboloid, while for other values of x and y, the z values inside S range from the bottom of the parabolic cylinder to the top of the parabolic cylinder. Thus, to evaluate I_z we need to write it as the sum of two integrals I_1 and I_2. These two regions are separated by a curve of the form $y = g(x)$ that is the projection on the xy plane of the intersection of the surfaces $z1$ and $z3$. Determine an equation for this curve in the form $y = g(x)$ and assign the name `g` to the Maple expression for $g(x)$.

2d. To see what the two regions look like, plot the curves $y = 0$, $y = f(x)$, and $y = g(x)$ for $-2 \le x \le 2$ on the same set of axes using the (two-dimensional) `plot` command with the `axes=BOXED` option.

2e. Let I_1 be the integral over the portion of S that projects into the region between $y = 0$ and $y = g(x)$. Identify the limits of integration on z in integral and write the Maple expression for this triple integral, giving it the name `I1`.

2f. Let I_2 be the integral over the portion of S that projects into the region between $y = g(x)$ and $y = f(x)$. Identify the limits of integration on z in integral and write the Maple expression for this triple integral, giving it the name `I2`.

2g. Now ask Maple to find the sum of I_1 and I_2. If the answer is not the same as the result of (1e), there is a mistake *somewhere* in your worksheet. Please correct it before submitting your report.

End 251 lab 4 (March 31, 2004).